# Introduction to Mobile Robotics

# Probabilistic Robotics

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# Probabilistic Robotics

**Key idea:**

**Explicit representation of uncertainty**

(using the calculus of probability theory)

- Perception = state estimation
- Action = utility optimization

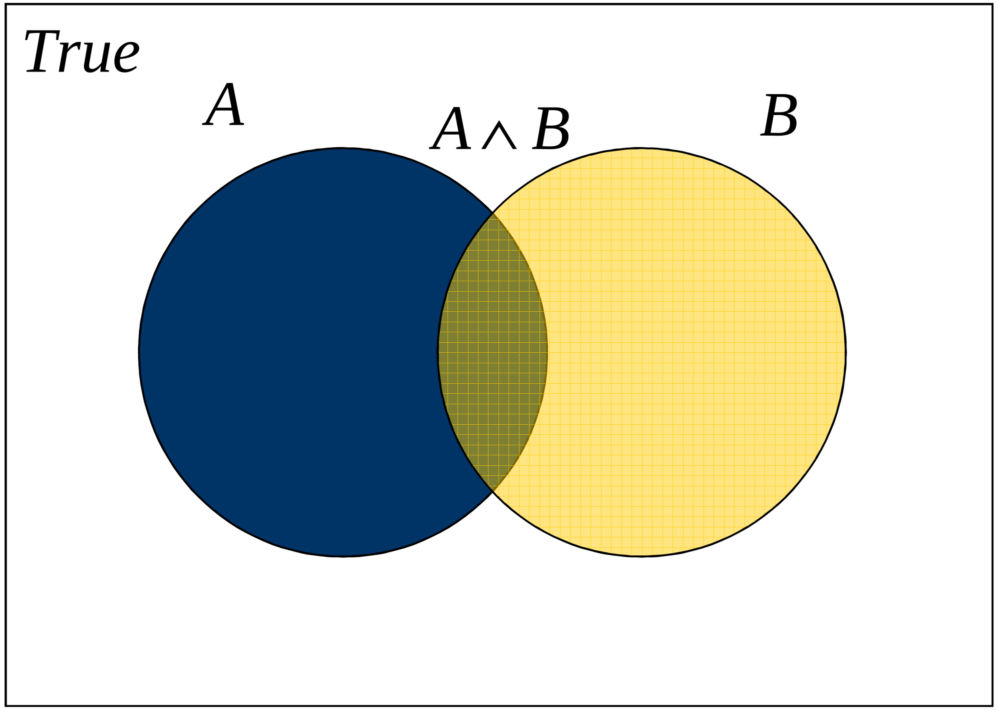
# Axioms of Probability Theory

Pr*(A)* denotes probability that proposition *A* is true.

- $0 \leq \Pr(A) \leq 1$
- $\Pr(True) = 1$ $\qquad\qquad$ $\Pr(False) = 0$
- $\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$

# A Closer Look at Axiom 3

$$\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$$



# Using the Axioms

$$
\begin{aligned}
\Pr(A \vee \neg A) &= \Pr(A) + \Pr(\neg A) - \Pr(A \wedge \neg A) \\
\Pr(True) &= \Pr(A) + \Pr(\neg A) - \Pr(False) \\
1 &= \Pr(A) + \Pr(\neg A) - 0 \\
\Pr(\neg A) &= 1 - \Pr(A)
\end{aligned}
$$

# Discrete Random Variables

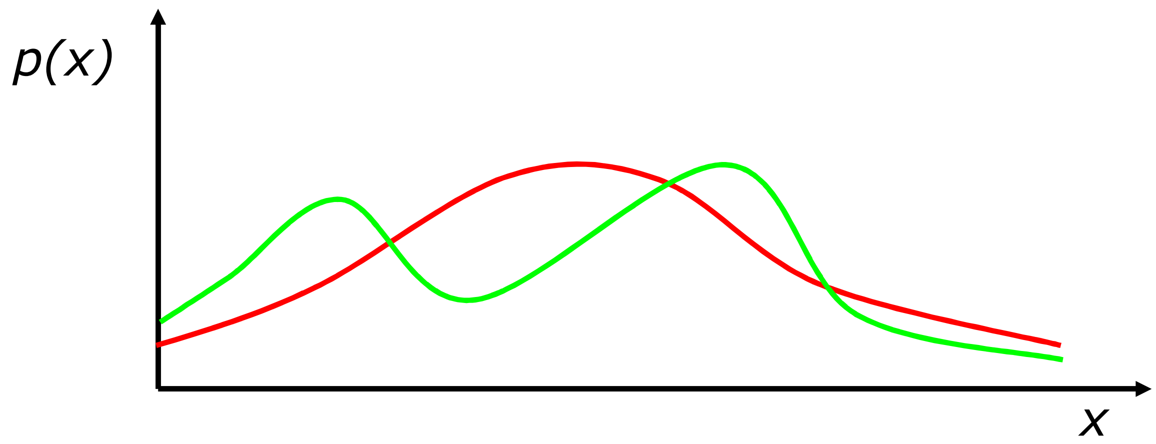
- $X$ denotes a random variable
- $X$ can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$
- $P(X=x_i)$ or $P(x_i)$ is the probability that the random variable $X$ takes on value $x_i$
- $P(\cdot)$ is called probability mass function
- E.g. $P(Room) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$

# Continuous Random Variables

- $X$ takes on values in the continuum.
- $p(X=x)$ or $p(x)$ is a probability density function

$$\Pr(x \in (a,b)) = \int_a^b p(x)dx$$

- E.g.



# “Probability Sums up to One”

**Discrete case**

$$\sum_x P(x) = 1$$

**Continuous case**

$$\int p(x)\, dx = 1$$

# Joint and Conditional Probability

- $P(X{=}x \text{ and } Y{=}y) = P(x,y)$

- If X and Y are independent then

$$P(x,y) = P(x)\,P(y)$$

- $P(x \mid y)$ is the probability of $x$ given $y$

$$P(x \mid y) = P(x,y) \,/\, P(y)$$
$$P(x,y) = P(x \mid y)\,P(y)$$

- If X and Y are independent then

$$P(x \mid y) = P(x)$$

# Law of Total Probability

**Discrete case**

$$P(x) = \sum_y P(x \mid y) P(y)$$

**Continuous case**

$$p(x) = \int p(x \mid y)\, p(y)\, dy$$

# Marginalization

**Discrete case**

$$P(x) = \sum_y P(x, y)$$

**Continuous case**

$$p(x) = \int p(x, y)\, dy$$

# Bayes Formula

$$P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x)$$

$$\Rightarrow$$

$$\boxed{P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}}$$

# Normalization

$$P(x \mid y) = \frac{P(y \mid x)\ P(x)}{P(y)} = \eta\ P(y \mid x)\, P(x)$$

$$\eta = P(y)^{-1} = \frac{1}{\sum_x P(y \mid x) P(x)}$$

**Algorithm:**

$$\forall x : \mathrm{aux}_{x|y} = P(y \mid x)\ P(x)$$

$$\eta = \frac{1}{\sum_x \mathrm{aux}_{x|y}}$$

$$\forall x : P(x \mid y) = \eta\ \mathrm{aux}_{x|y}$$

# Bayes Rule with Background Knowledge

$$P(x \mid y, z) = \frac{P(y \mid x, z)\ P(x \mid z)}{P(y \mid z)}$$

# Conditional Independence

$$P(x, y \mid z) = P(x \mid z) P(y \mid z)$$

- Equivalent to $P(x \mid z) = P(x \mid z, y)$

  and $P(y \mid z) = P(y \mid z, x)$

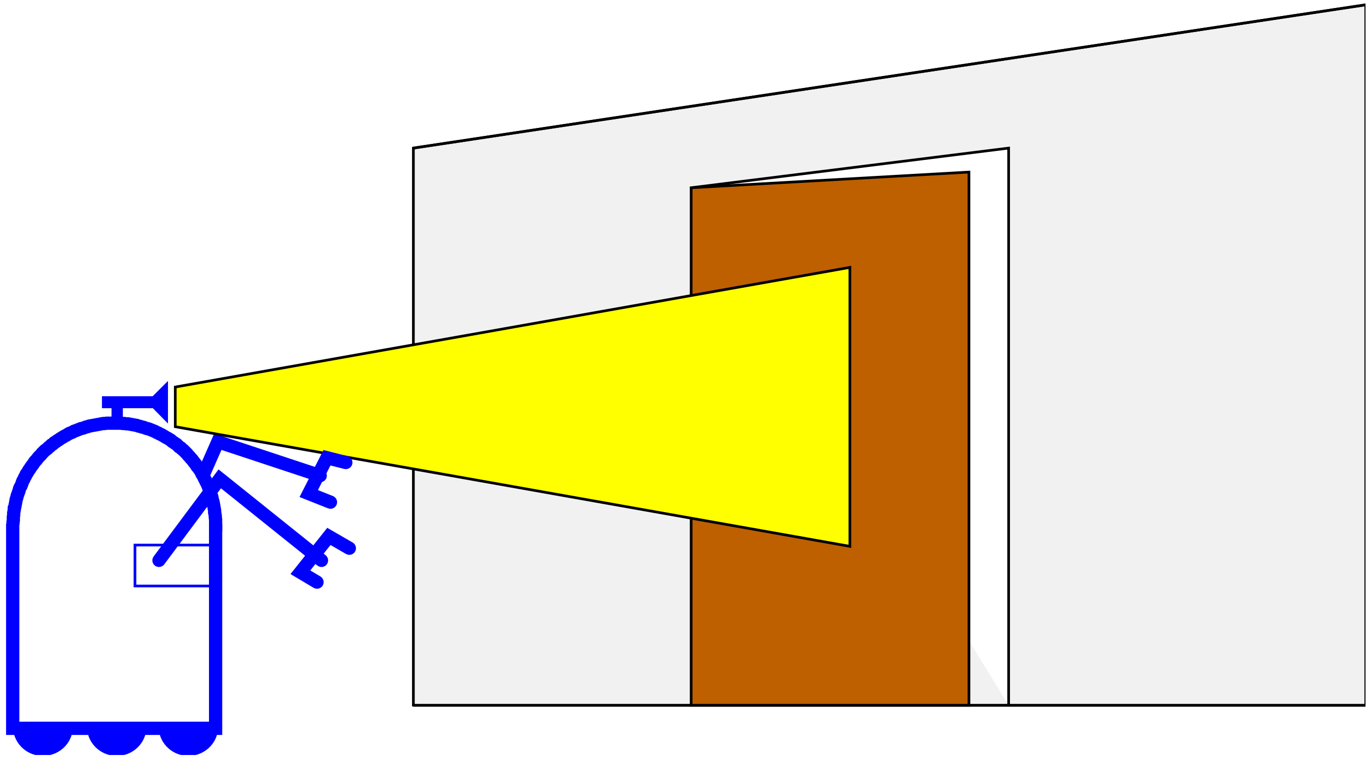
- But this does not necessarily mean

$$P(x, y) = P(x) P(y)$$

(independence/marginal independence)

# Simple Example of State Estimation

- Suppose a robot obtains measurement z
- What is $P(open|z)$?



# Causal vs. Diagnostic Reasoning

- $P(open|z)$ is diagnostic
- $P(z|open)$ is causal
- Often causal knowledge is easier to obtain

**count frequencies!**

- Bayes rule allows us to use causal knowledge:

$$P(open \mid z) = \frac{P(z \mid open) P(open)}{P(z)}$$

# Example

- $P(z|open) = 0.6$ $\qquad$ $P(z|\neg open) = 0.3$
- $P(open) = P(\neg open) = 0.5$

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z \mid open)\,p(open) + P(z \mid \neg open)\,p(\neg open)}$$

$$P(open \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{0.3}{0.3 + 0.15} = 0.67$$

- *z* raises the probability that the door is open

# Combining Evidence

- Suppose our robot obtains another observation $z_2$
- How can we integrate this new information?
- More generally, how can we estimate $P(x| z_1...z_n)$?

# Recursive Bayesian Updating

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x, z_1, \ldots, z_{n-1})\, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}$$

**Markov assumption:**
$z_n$ is independent of $z_1, \ldots, z_{n-1}$ if we know $x$

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x)\, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}$$

$$= \eta\, P(z_n \mid x)\, P(x \mid z_1, \ldots, z_{n-1})$$

$$= \eta_{1\ldots n} \prod_{i=1\ldots n} P(z_i \mid x)\, P(x)$$

# Example: Second Measurement

- $P(z_2|open) = 0.5$ $\qquad$ $P(z_2|\neg open) = 0.6$
- $P(open|z_1)=2/3$

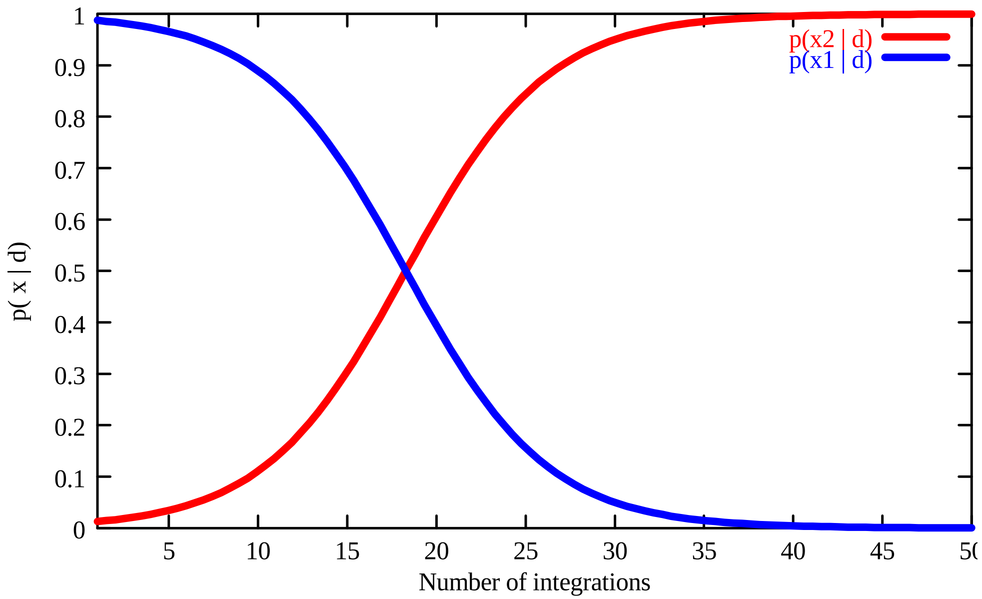
$$P(open \mid z_2, z_1) = \frac{P(z_2 \mid open)\, P(open \mid z_1)}{P(z_2 \mid open)\, P(open \mid z_1) + P(z_2 \mid \neg open)\, P(\neg open \mid z_1)}$$

$$= \frac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{1}{2} \cdot \frac{2}{3} + \frac{3}{5} \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{5}} = \frac{\frac{1}{3}}{\frac{8}{15}} = \frac{5}{8} = 0.625$$

- $z_2$ lowers the probability that the door is open

# A Typical Pitfall

- Two possible locations $x_1$ and $x_2$
- $P(x_1)=0.99$
- $P(z|x_2)=0.09$ $P(z|x_1)=0.07$



# Actions

- Often the world is **dynamic** since
  - **actions carried out by the robot**,
  - **actions carried out by other agents**,
  - or just the **time** passing by

  change the world

- How can we **incorporate** such **actions**?

## Typical Actions

- The robot **turns its wheels** to move
- The robot **uses its manipulator** to grasp an object
- Plants grow over **time**…

- Actions are **never carried out with absolute certainty**
- In contrast to measurements, **actions generally increase the uncertainty**

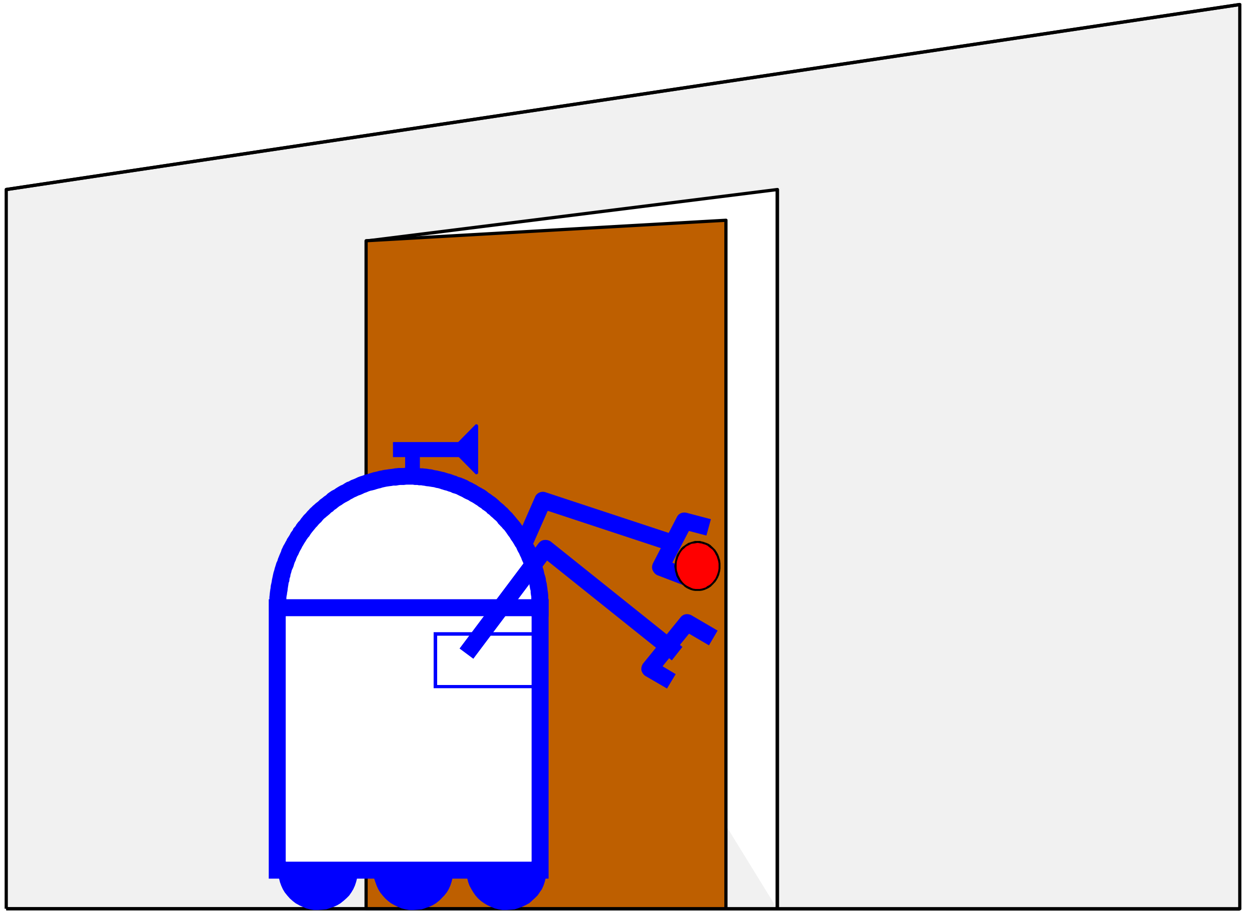
## Modeling Actions

- To incorporate the outcome of an action *u* into the current "belief", we use the conditional pdf

$$P(x \mid u, x')$$

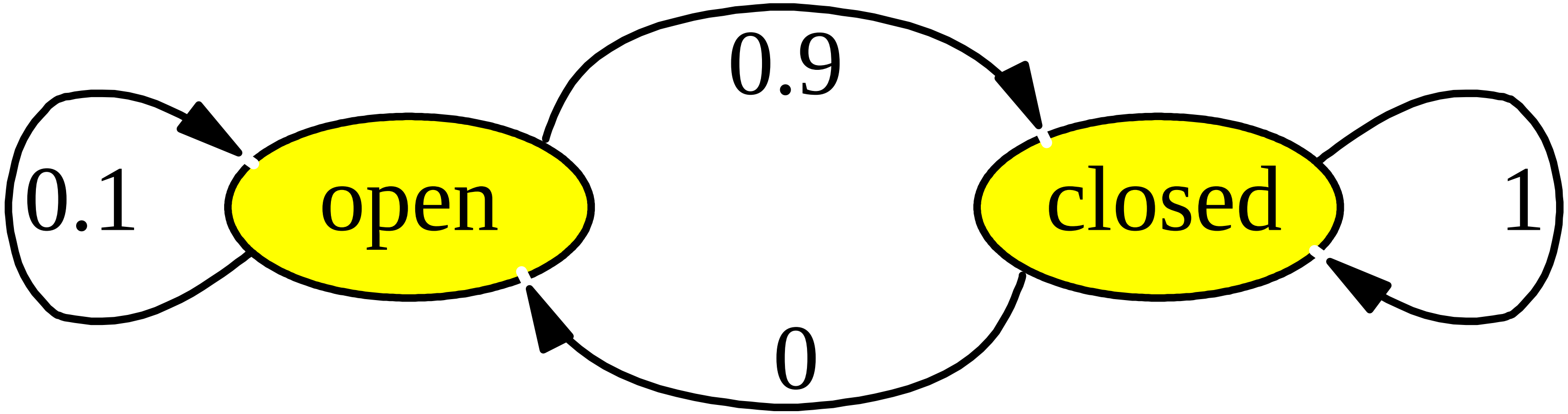
- This term specifies the pdf that **executing *u* changes the state from *x'* to *x*.**

# Example: Closing the door



# State Transitions

$P(x|u,x')$ for $u$ = “close door”:




If the door is open, the action “close door” succeeds in 90% of all cases

# Integrating the Outcome of Actions

Continuous case:

$$P(x \mid u) = \int P(x \mid u, x')P(x')dx'$$

Discrete case:

$$P(x \mid u) = \sum P(x \mid u, x')P(x')$$

# Example: The Resulting Belief

$$
\begin{aligned}
P(closed \mid u) &= \sum P(closed \mid u, x')P(x') \\
&= P(closed \mid u, open)P(open) \\
&\quad + P(closed \mid u, closed)P(closed) \\
&= \frac{9}{10} * \frac{5}{8} + \frac{1}{1} * \frac{3}{8} = \frac{15}{16}
\end{aligned}
$$

$$
\begin{aligned}
P(open \mid u) &= \sum P(open \mid u, x')P(x') \\
&= P(open \mid u, open)P(open) \\
&\quad + P(open \mid u, closed)P(closed) \\
&= \frac{1}{10} * \frac{5}{8} + \frac{0}{1} * \frac{3}{8} = \frac{1}{16} \\
&= 1 - P(closed \mid u)
\end{aligned}
$$

# Bayes Filters: Framework

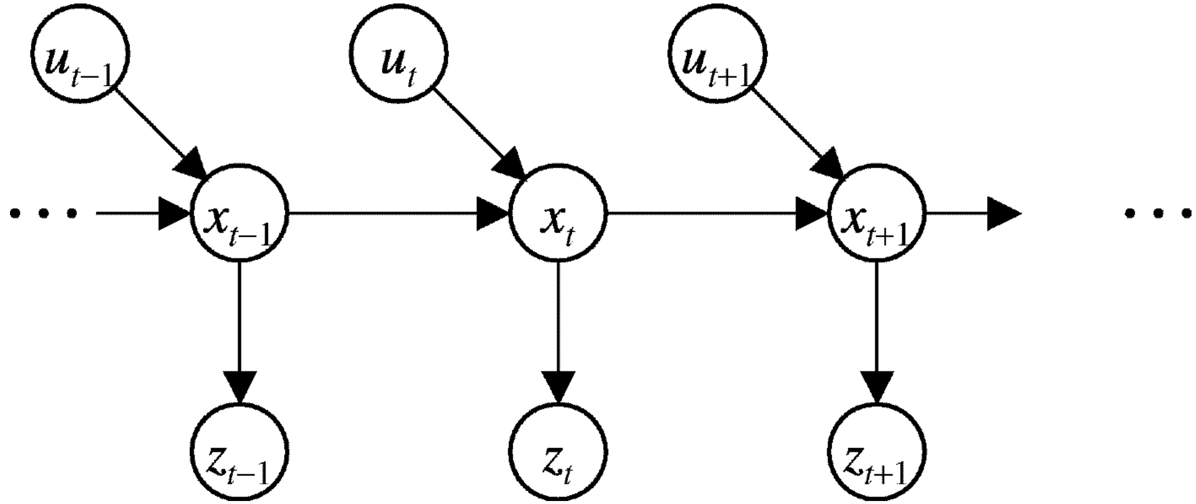
- **Given:**
  - Stream of observations *z* and action data *u:*

$$d_t = \{u_1, z_1 \ldots, u_t, z_t\}$$

  - Sensor model *P(z|x)*
  - Action model *P(x|u,x')*
  - Prior probability of the system state *P(x)*
- **Wanted:**
  - Estimate of the state *X* of a dynamical system
  - The posterior of the state is also called **Belief**:

$$Bel(x_t) = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)$$

# Markov Assumption




$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$$

$$p(x_t \mid x_{1:t-1}, z_{1:t}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

**Underlying Assumptions**

- Static world
- Independent noise
- Perfect model, no approximation errors

# Bayes Filters

$z$ = observation
$u$ = action
$x$ = state

$$\boxed{Bel(x_t)} = P(x_t \mid u_1, z_1, \ldots, u_t, z_t)$$

**Bayes**
$$= \eta P(z_t \mid x_t, u_1, z_1, \ldots, u_t) P(x_t \mid u_1, z_1, \ldots, u_t)$$

**Markov**
$$= \eta P(z_t \mid x_t) P(x_t \mid u_1, z_1, \ldots, u_t)$$

**Total prob.**
$$= \eta P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots, u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \ldots, u_t) dx_{t-1}$$

**Markov**
$$= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \ldots, u_t) dx_{t-1}$$

**Markov**
$$= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1}) dx_{t-1}$$

$$\boxed{= \eta\, P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\, Bel(x_{t-1})\, dx_{t-1}}$$

$$Bel(x_t) = \eta\, P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\, Bel(x_{t-1})\, dx_{t-1}$$

1. Algorithm **Bayes_filter**( *Bel(x),d* ):
2. $\eta$=0
3. If *d* is a perceptual data item *z* then
4. For all *x* do
5. $Bel'(x) = P(z \mid x) Bel(x)$
6. $\eta = \eta + Bel'(x)$
7. For all *x* do
8. $Bel'(x) = \eta^{-1} Bel'(x)$
9. Else if *d* is an action data item *u* then
10. For all *x* do
11. $Bel'(x) = \int P(x \mid u, x')\, Bel(x')\, dx'$
12. Return *Bel'(x)*

# Bayes Filters are Familiar!

$$Bel(x_t) = \eta \, P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \, Bel(x_{t-1}) \, dx_{t-1}$$

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)

# Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.