Introduction to Mobile Robotics

Bayes Filter – Kalman Filter

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Bayes Filter Reminder

$$bel(x_t) = \eta\ p(z_t \mid x_t) \int p(x_t \mid u_t, x_{t-1})\ bel(x_{t-1})\ dx_{t-1}$$

- Prediction

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\ bel(x_{t-1})\ dx_{t-1}$$

- Correction

$$bel(x_t) = \eta\ p(z_t \mid x_t)\,\overline{bel}(x_t)$$

Kalman Filter

- Bayes filter with **Gaussians**
- Developed in the late 1950's
- Most relevant Bayes filter variant in practice
- Applications range from economics, weather forecasting, satellite navigation to robotics and many more.

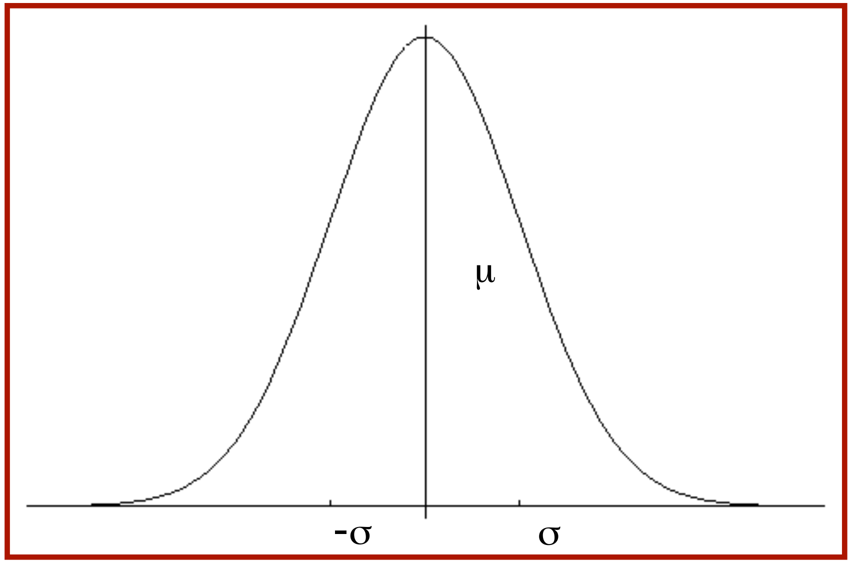
- The Kalman filter "algorithm" is a couple of **matrix multiplications**!

Gaussians

$$p(x) \sim N(\mu, \sigma^2):$$

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

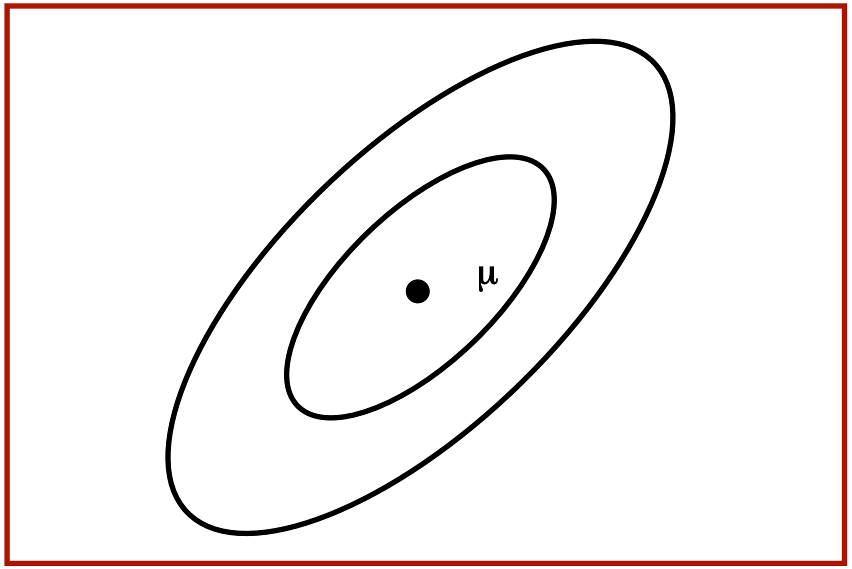
Univariate



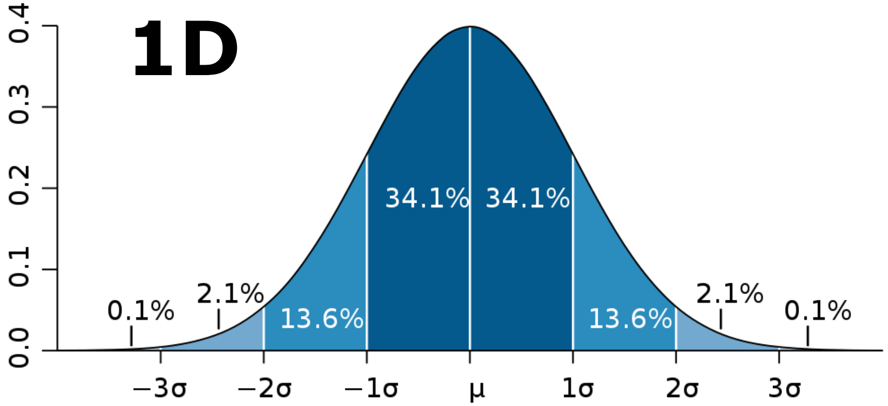
$$p(\mathbf{x}) \sim N(\mu, \Sigma):$$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^t \Sigma^{-1}(\mathbf{x}-\mu)}$$

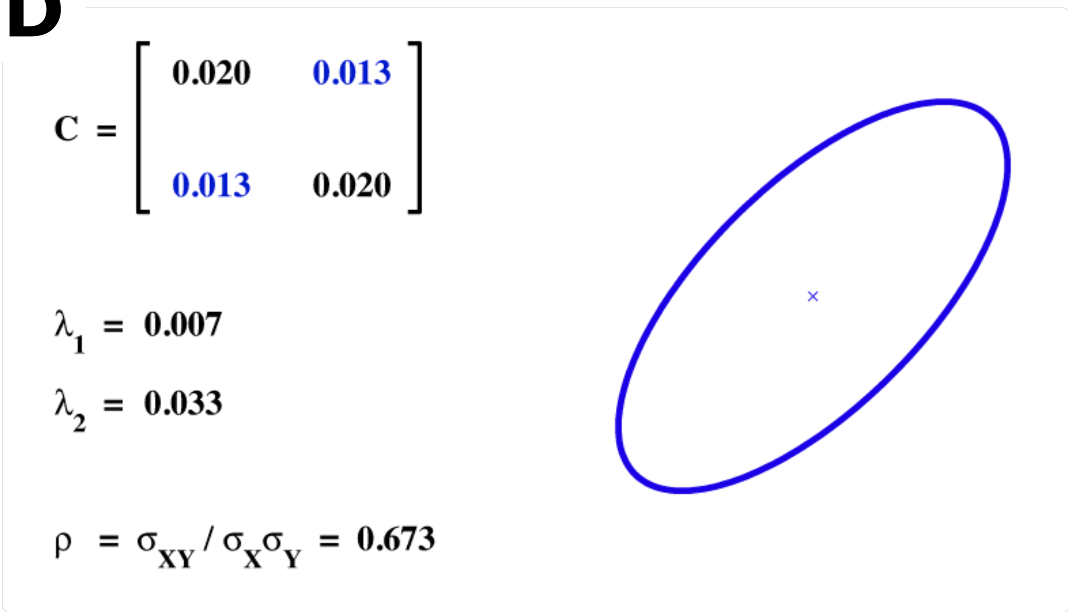
Multivariate



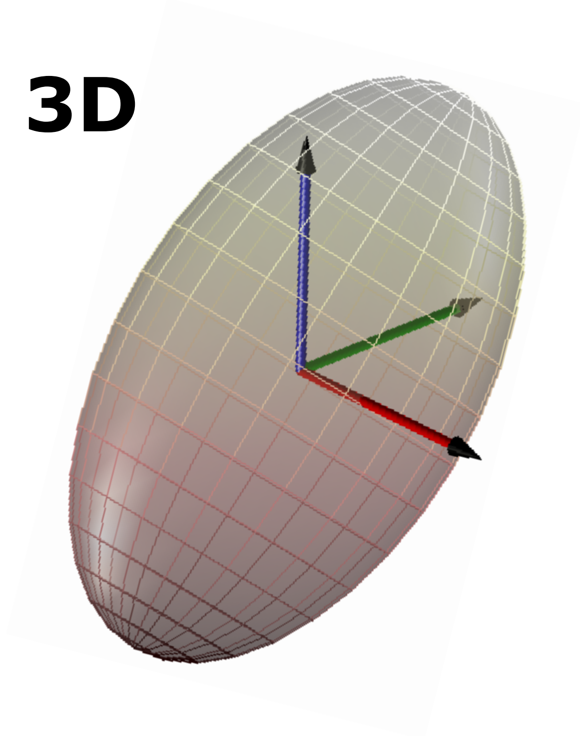
Gaussians



2D



3D



Properties of Gaussians

- Univariate case

$$\left.\begin{array}{l} X \sim N(\mu, \sigma^2) \\ Y = aX + b \end{array}\right\} \Rightarrow \quad Y \sim N(a\mu + b, a^2\sigma^2)$$

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \sigma_1{}^2) \\ X_2 \sim N(\mu_2, \sigma_2{}^2) \end{array}\right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\sigma_2{}^2}{\sigma_1{}^2 + \sigma_2{}^2}\mu_1 + \frac{\sigma_1{}^2}{\sigma_1{}^2 + \sigma_2{}^2}\mu_2, \quad \frac{1}{\sigma_1{}^{-2} + \sigma_2{}^{-2}} \right)$$

Properties of Gaussians

- Multivariate case

$$\left.\begin{aligned} X &\sim N(\mu,\Sigma) \\ Y &= AX + B \end{aligned}\right\} \quad \Rightarrow \quad Y \sim N(A\mu + B, A\Sigma A^T)$$

$$\left.\begin{aligned} X_1 &\sim N(\mu_1,\Sigma_1) \\ X_2 &\sim N(\mu_2,\Sigma_2) \end{aligned}\right\} \Rightarrow p(X_1)\cdot p(X_2) \sim N\left(\frac{\Sigma_2}{\Sigma_1+\Sigma_2}\mu_1 + \frac{\Sigma_1}{\Sigma_1+\Sigma_2}\mu_2, \quad \frac{1}{\Sigma_1^{-1}+\Sigma_2^{-1}} \right)$$

(where division "–" denotes matrix inversion)

- We **stay Gaussian** as long as we start with Gaussians and perform only **linear transformations**

Discrete Kalman Filter

Estimates the state x of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

with a measurement

$$z_t = C_t x_t + \delta_t$$

Components of a Kalman Filter

A_t Matrix (nxn) that describes how the state evolves from *t*-1 to *t* without controls or noise.

B_t Matrix (nxl) that describes how the control u_t changes the state from *t*-1 to t.

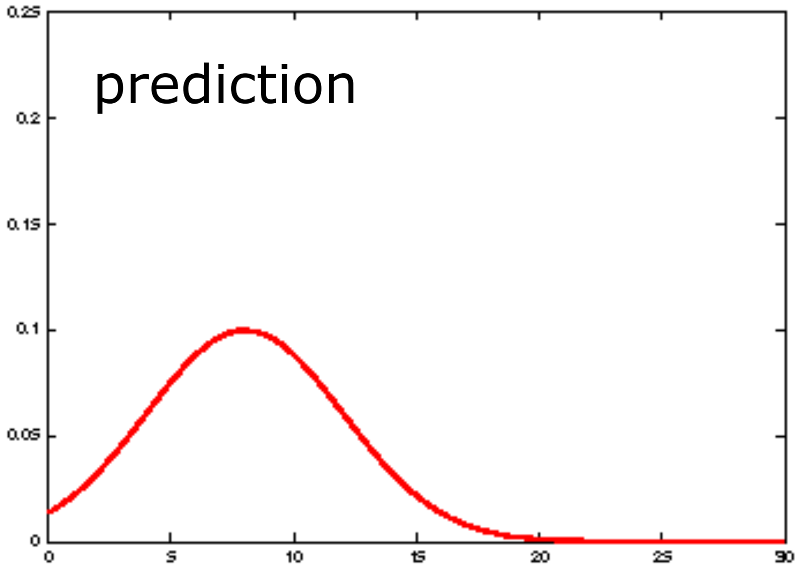
C_t Matrix (kxn) that describes how to map the state x_t to an observation z_t.

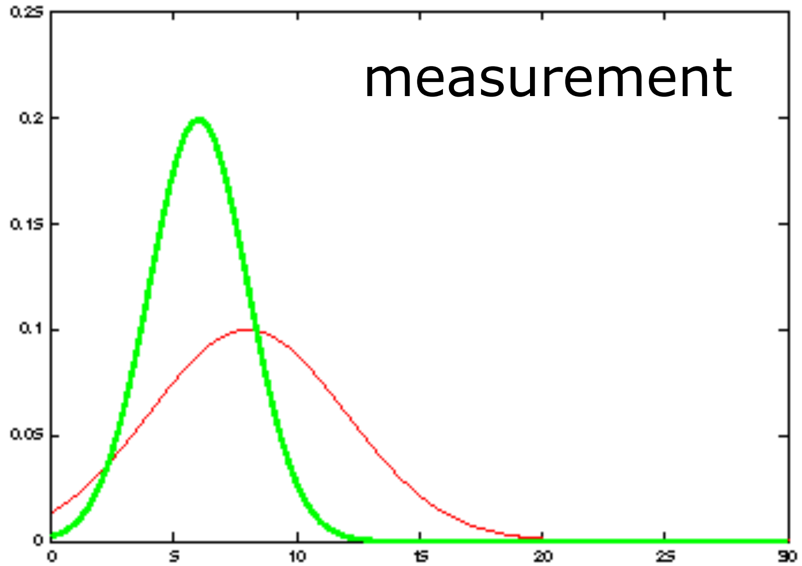
ε_t

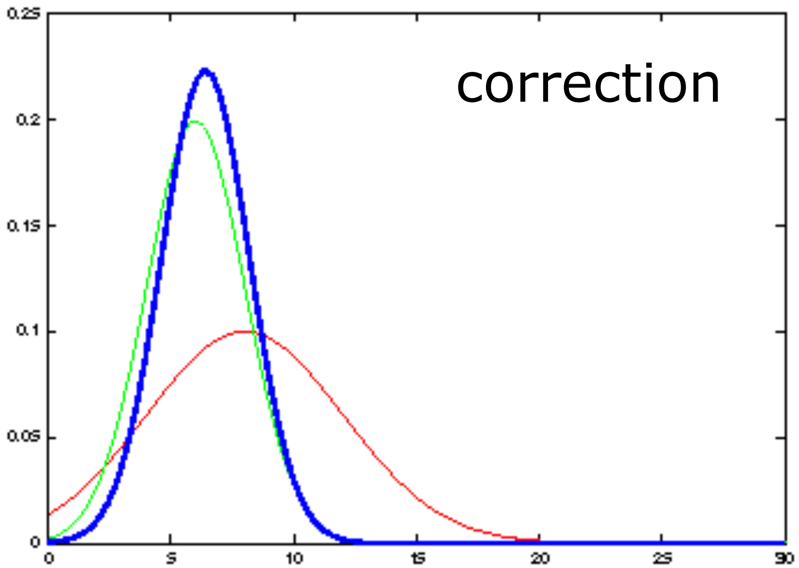
δ_t

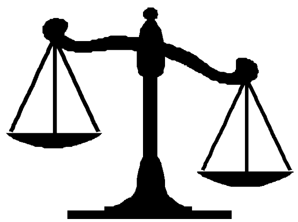
Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance Q_t and R_t respectively.

Kalman Filter Updates in 1D



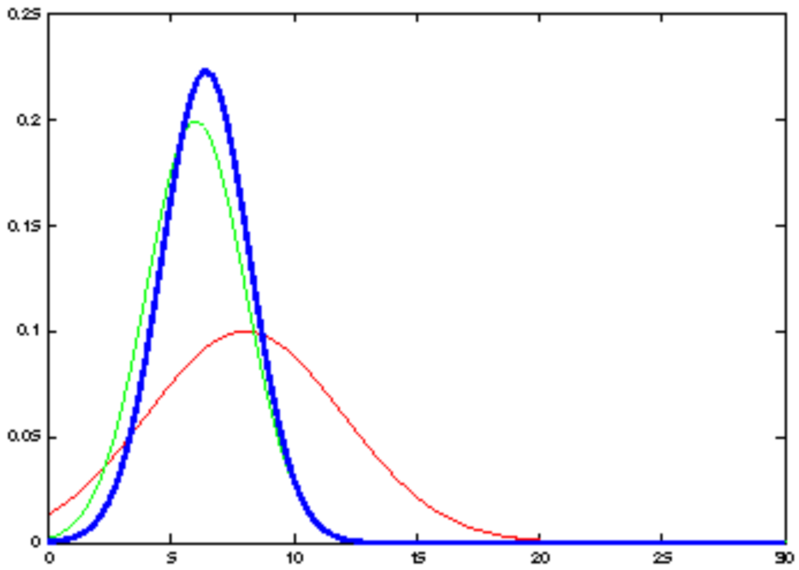






It's a weighted mean!

Kalman Filter Updates in 1D

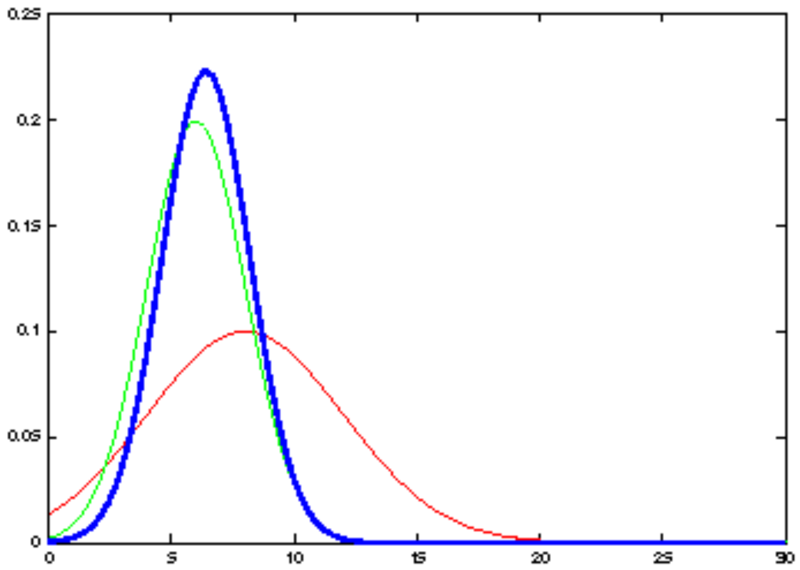


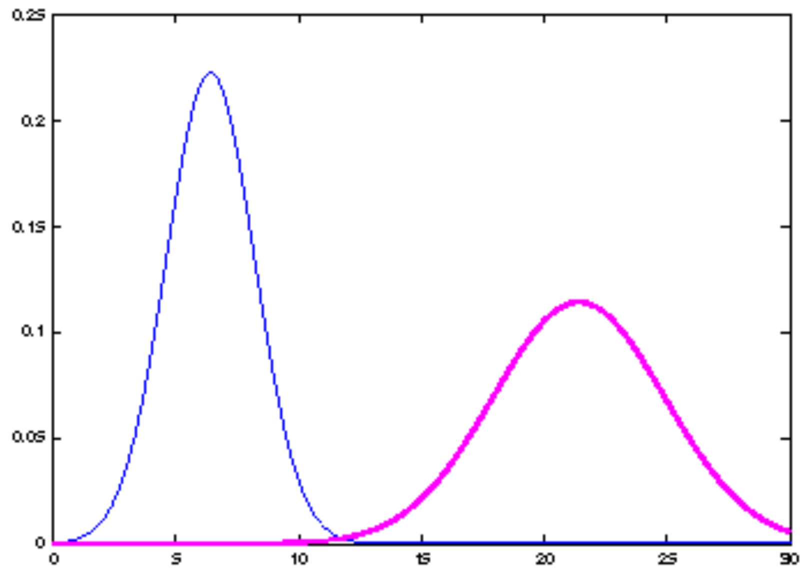
How to get the blue one?
Kalman correction step

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases} \quad \text{with} \quad K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \quad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$$

Kalman Filter Updates in 1D





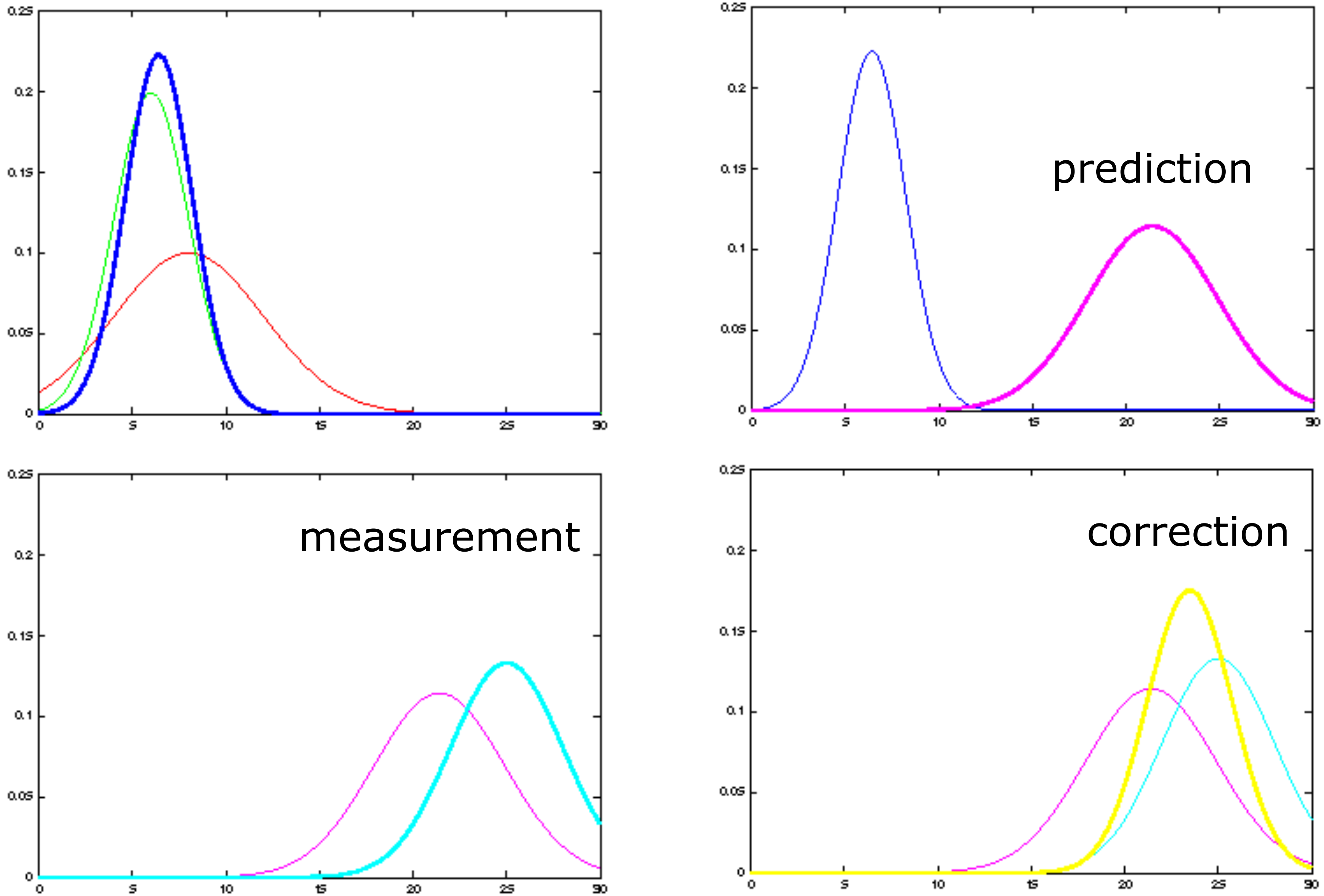
$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t \end{cases}$$

How to get the magenta one?

State prediction step

Kalman Filter Updates



Linear Gaussian Systems: Initialization

Initial belief is normally distributed:

$$bel(x_0) = N\left(x_0; \mu_0, \Sigma_0\right)$$

Linear Gaussian Systems: Dynamics

Dynamics are linear functions of the state and the control plus additive noise:

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

$$p(x_t \mid u_t, x_{t-1}) = N\left(x_t; A_t x_{t-1} + B_t u_t, Q_t\right)$$

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \qquad bel(x_{t-1})\, dx_{t-1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(x_t; A_t x_{t-1} + B_t u_t, Q_t\right) \quad \sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

Linear Gaussian Systems: Dynamics

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \qquad bel(x_{t-1})\, dx_{t-1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(x_t; A_t x_{t-1} + B_t u_t, Q_t\right) \quad \sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

$$\Downarrow$$

$$\overline{bel}(x_t) = \eta \int \exp\left\{-\frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T Q_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)\right\}$$

$$\exp\left\{-\frac{1}{2}(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})\right\} dx_{t-1}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t \end{cases}$$

Linear Gaussian Systems: Observations

Observations is a linear function of the state plus additive noise:

$$z_t = C_t x_t + \delta_t$$

$$p(z_t \mid x_t) = N\left(z_t; C_t x_t, R_t\right)$$

$$bel(x_t) = \eta \quad p(z_t \mid x_t) \quad \overline{bel}(x_t)$$

$$\Downarrow \qquad \Downarrow$$

$$\sim N\left(z_t; C_t x_t, R_t\right) \qquad \sim N\left(x_t; \bar{\mu}_t, \bar{\Sigma}_t\right)$$

Linear Gaussian Systems: Observations

$$bel(x_t) = \eta \quad p(z_t \mid x_t) \qquad \overline{bel}(x_t)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(z_t; C_t x_t, R_t\right) \qquad \sim N\left(x_t; \bar{\mu}_t, \bar{\Sigma}_t\right)$$

$$\Downarrow$$

$$bel(x_t) = \eta \exp\left\{-\frac{1}{2}(z_t - C_t x_t)^T R_t^{-1}(z_t - C_t x_t)\right\} \exp\left\{-\frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)\right\}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \qquad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$$

Kalman Filter Algorithm

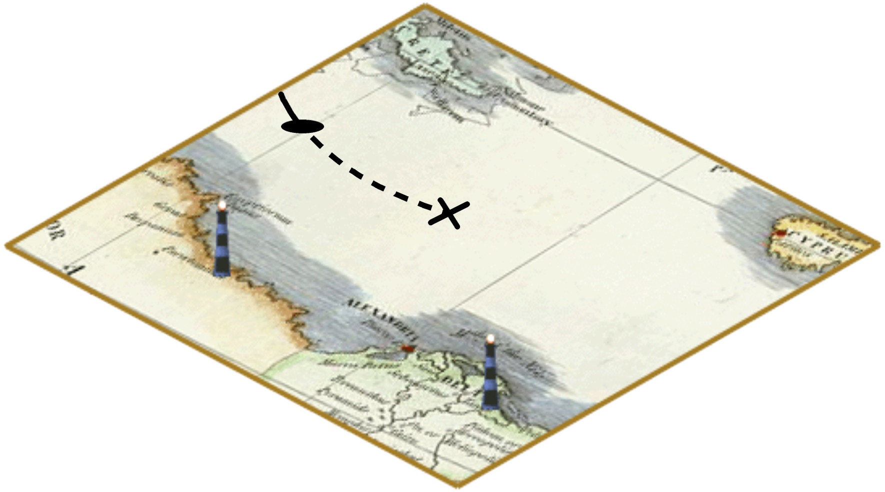
1. Algorithm **Kalman_filter**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

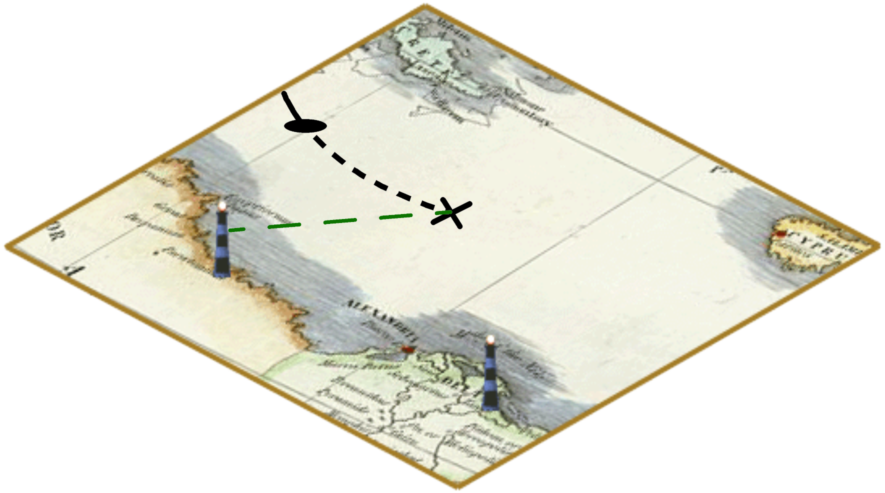
2. Prediction:
3. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t$

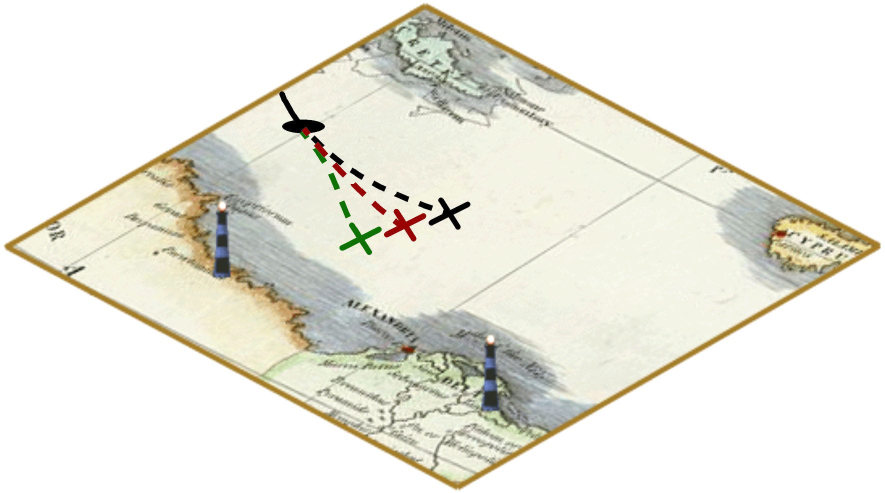
5. Correction:
6. $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

Kalman Filter Algorithm

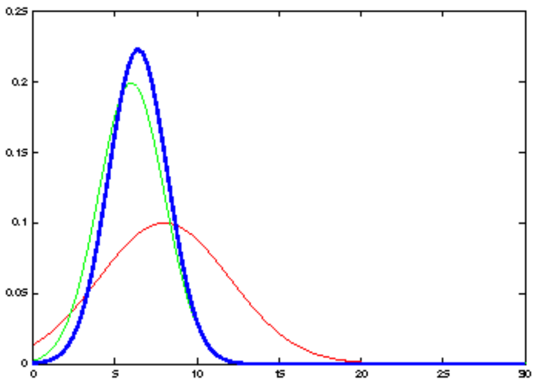






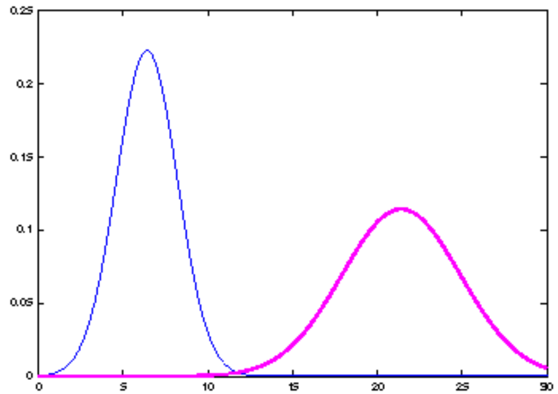
The Prediction-Correction-Cycle

Prediction

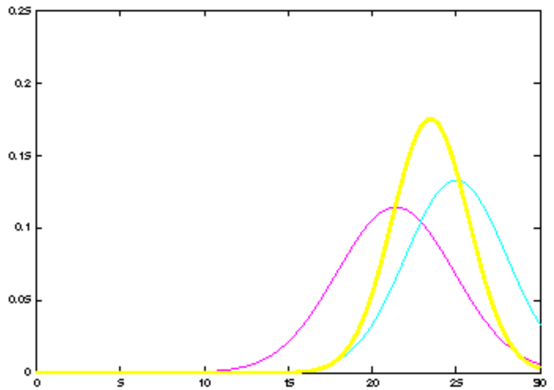


$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t \end{cases}$$

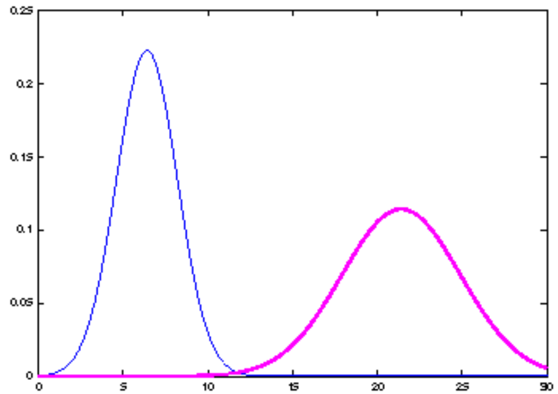


The Prediction-Correction-Cycle



$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases}, \quad K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$$



Correction

The Prediction-Correction-Cycle

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases}, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

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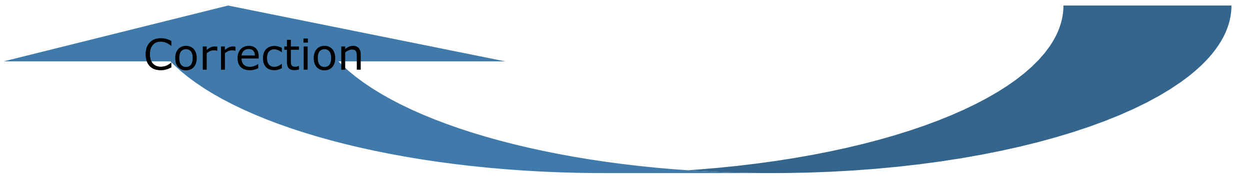


Kalman Filter Summary

- Only two parameters describe belief about the state of the system
- **Highly efficient:** Polynomial in the measurement dimensionality *k* and state dimensionality *n*:

$$O(k^{2.376} + n^2)$$

- **Optimal for linear Gaussian systems**!
- However: Most robotics systems are **nonlinear**!
- Can only model unimodal beliefs

Nonlinear Dynamic Systems

- Most realistic robotic problems involve nonlinear functions

$$x_t = g(u_t, x_{t-1})$$

$$z_t = h(x_t)$$

EKF Linearization: First Order Taylor Series Expansion

- Prediction:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \ (x_{t-1} - \mu_{t-1})$$

- Correction:

$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

$$h(x_t) \approx h(\bar{\mu}_t) + H_t \ (x_t - \bar{\mu}_t)$$

EKF Algorithm

1. **Extended_Kalman_filter**(μ_{t-1}, Σ_{t-1}, u_t, z_t):

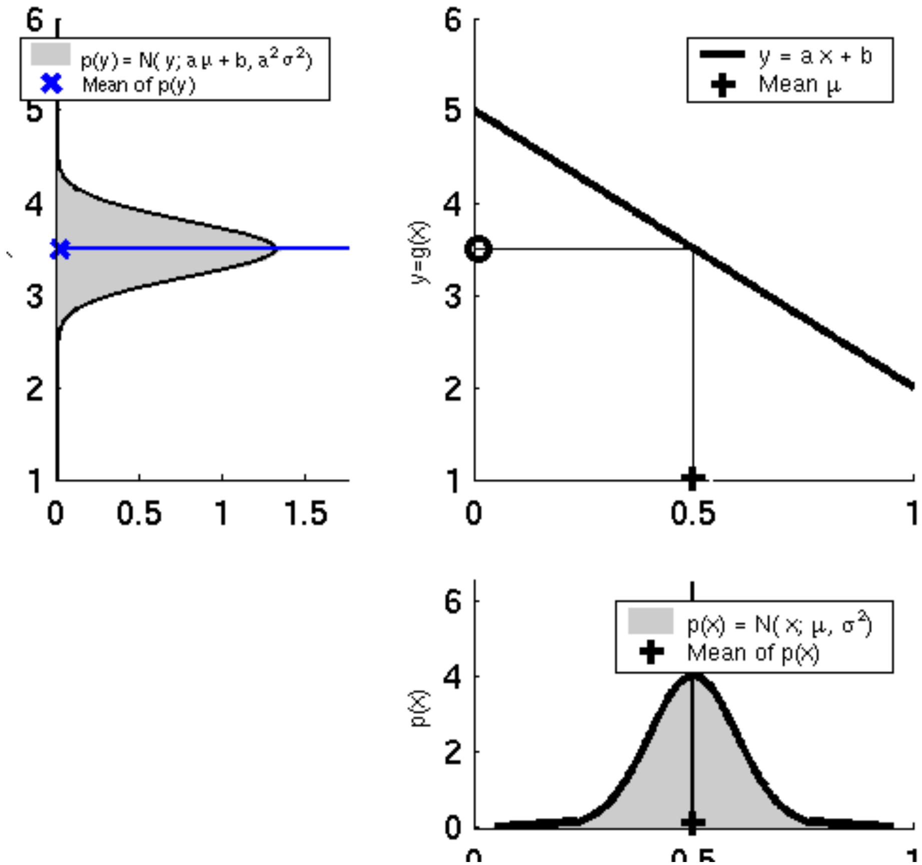
2. Prediction:
3. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ ⟵ $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + Q_t$ ⟵ $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t$

5. Correction:
6. $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + R_t)^{-1}$ ⟵ $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t))$ ⟵ $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$ ⟵ $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

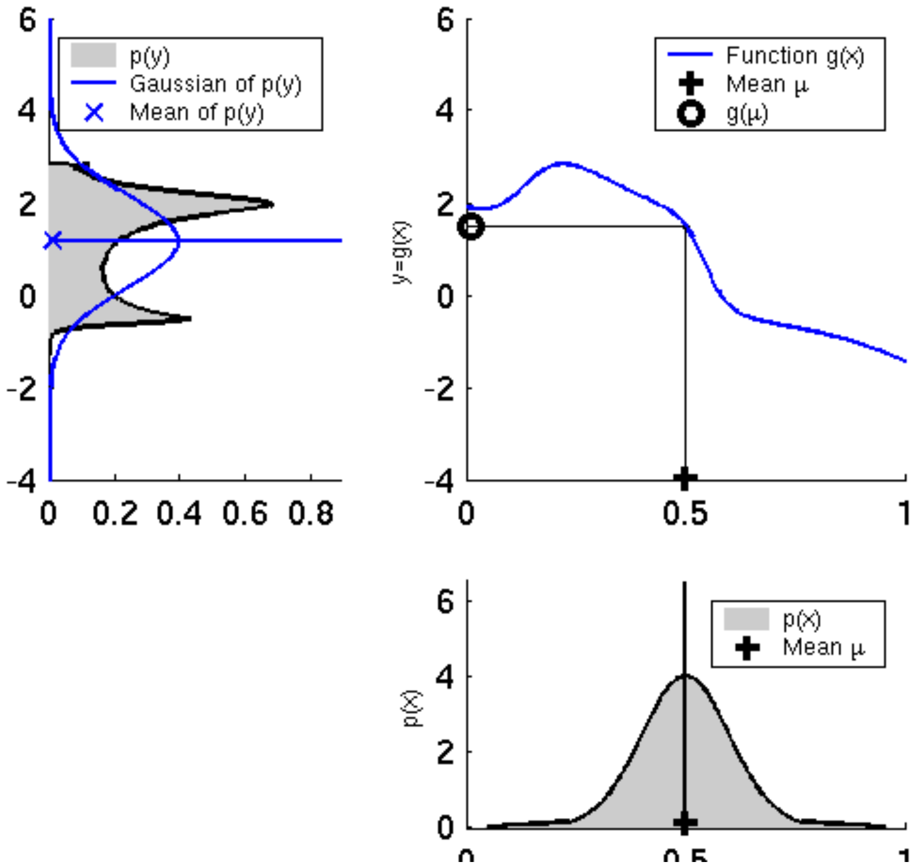
9. Return μ_t, Σ_t

$$H_t = \frac{\partial h(\bar{\mu}_t)}{\partial x_t} \qquad G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}}$$

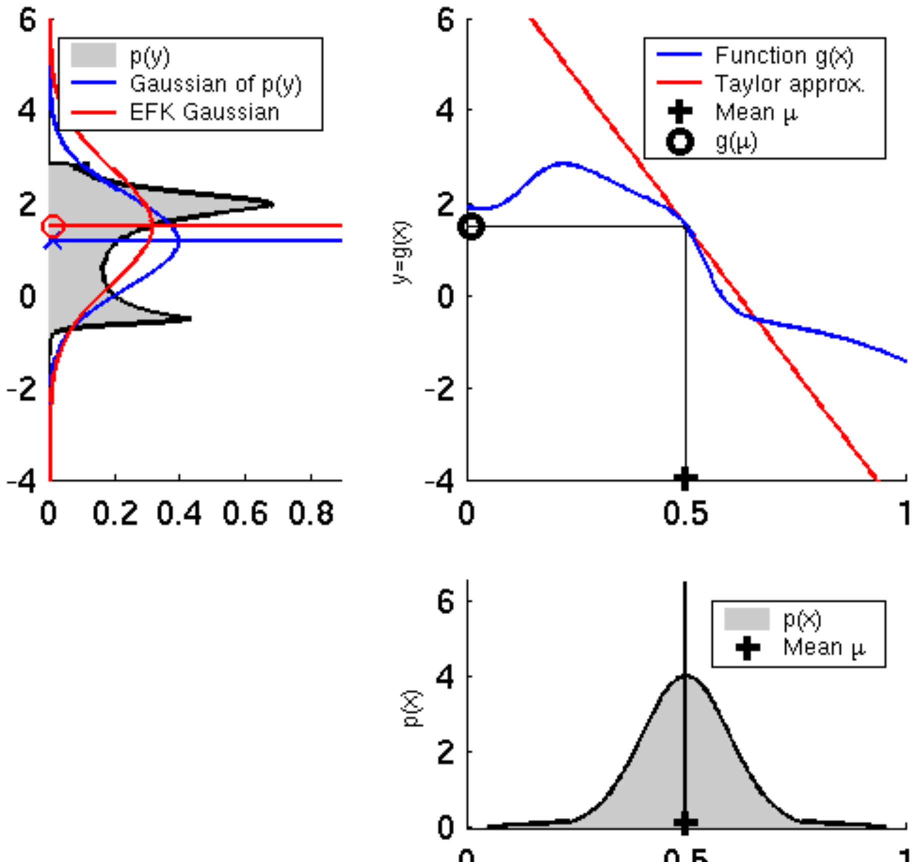
Linearity Assumption Revisited



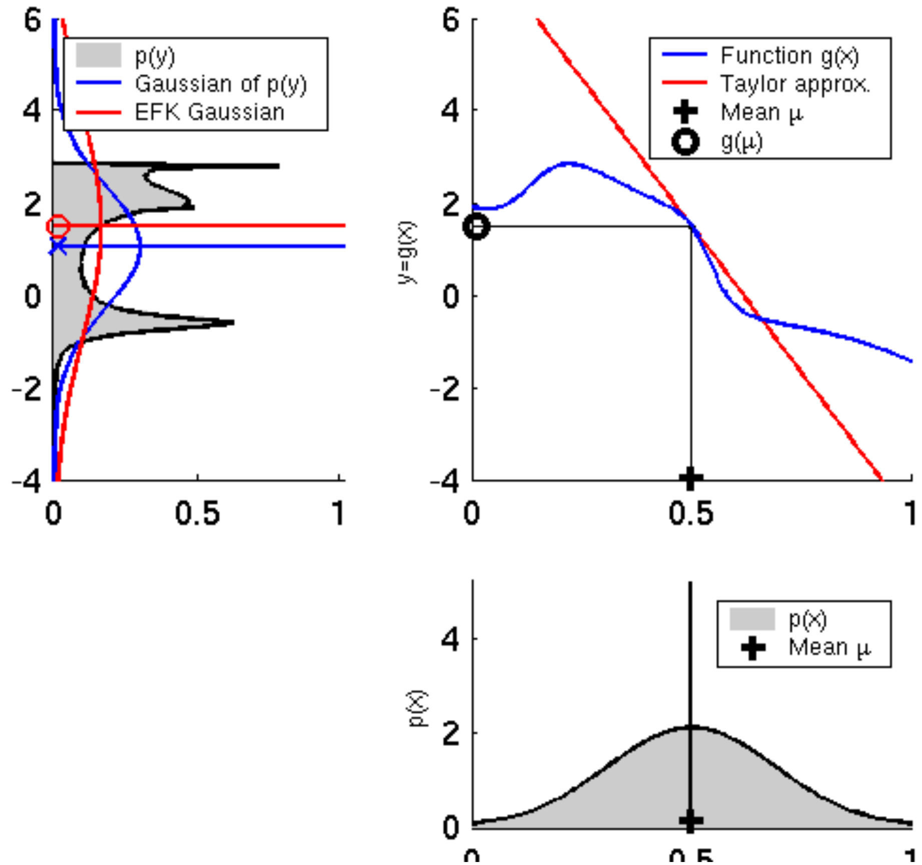
Non-linear Function



EKF Linearization (1)



EKF Linearization (2)



EKF Linearization (3)