Introduction to Mobile Robotics

Compact Course on Linear Algebra

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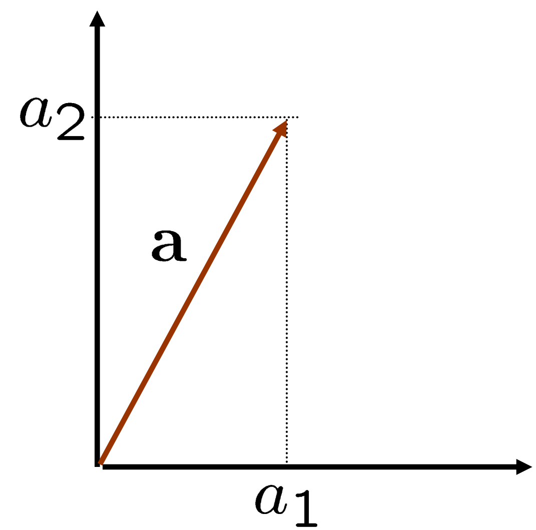
Vectors

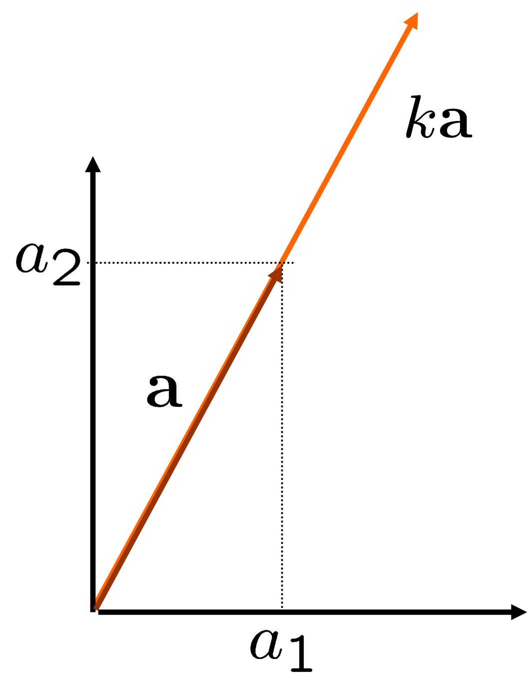
- Arrays of numbers
- Vectors represent a point in a *n* dimensional space

$$(a_1) \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$



Vectors: Scalar Product

- Scalar-Vector Product $k\mathbf{a}$
- Changes the length of the vector, but **not** its direction

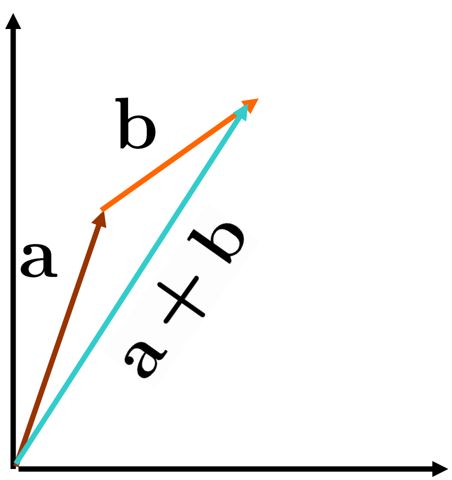


Vectors: Sum

- Sum of vectors (is commutative)

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

- Can be visualized as “chaining” the vectors.

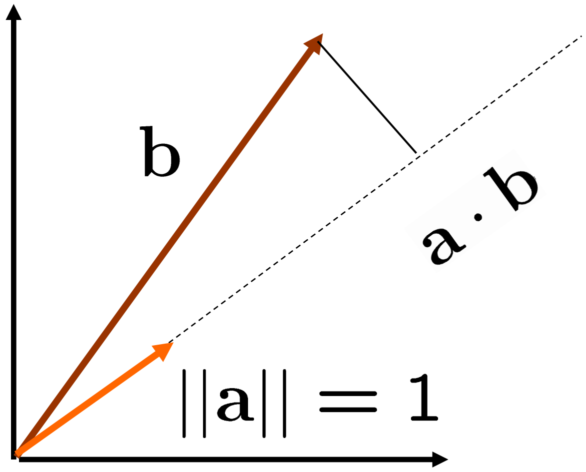


Vectors: Dot Product

- Inner product of vectors (is a scalar)

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = \sum_i a_i b_i$$

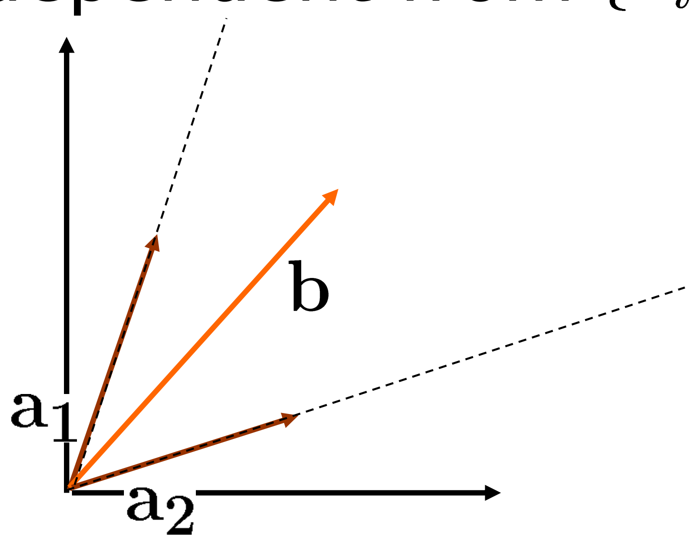
- If one of the two vectors, e.g. $\mathbf{a}$, has $||\mathbf{a}|| = 1$ the inner product $\mathbf{a} \cdot \mathbf{b}$ returns the length of the projection of $\mathbf{b}$ along the direction of $\mathbf{a}$



- If $\mathbf{a} \cdot \mathbf{b} = 0$, the two vectors are **orthogonal**

Vectors: Linear (In)Dependence

- A vector $\mathbf{b}$ is **linearly dependent** from $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ if $\mathbf{b} = \sum_i k_i \mathbf{a}_i$
- In other words, if $\mathbf{b}$ can be obtained by summing up the $\mathbf{a}_i$ properly scaled
- If there exist no $\{k_i\}$ such that $\mathbf{b} = \sum_i k_i \mathbf{a}_i$ then $\mathbf{b}$ is independent from $\{\mathbf{a}_i\}$



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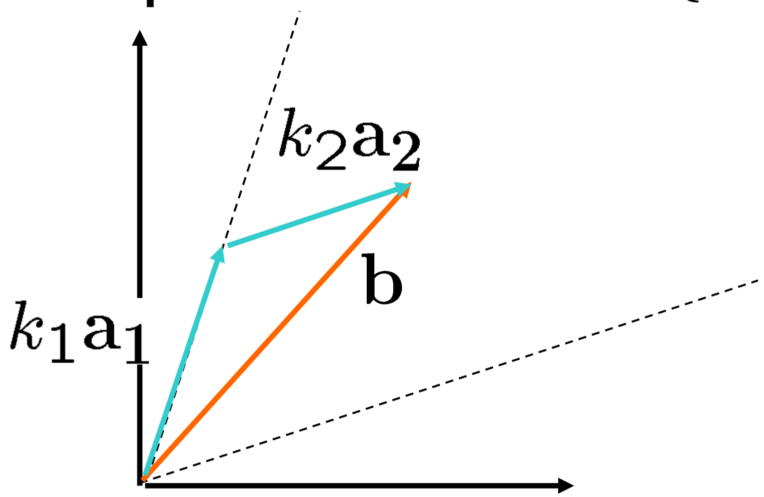


Matrices

- A matrix is written as a table of values

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \qquad A : \underset{\substack{\uparrow \\ \text{rows}}}{n} \times \underset{\substack{\uparrow \\ \text{columns}}}{m}$$

- **1st index** refers to the **row**
- **2nd index** refers to the **column**
- Note: a d-dimensional vector is equivalent to a dx1 matrix

Matrices as Collections of Vectors

- Column vectors

$$\begin{pmatrix} \mathbf{a}_{*1} & \mathbf{a}_{*2} & \cdots & \mathbf{a}_{*m} \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

Matrices as Collections of Vectors

- Row vectors

$$\mathbf{A} = \begin{pmatrix} \boxed{a_{11} \quad a_{12} \quad \cdots \quad a_{1m}} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2m}} \\ \vdots \\ \boxed{a_{n1} \quad a_{n2} \quad \cdots \quad a_{nm}} \end{pmatrix} \qquad \begin{pmatrix} \mathbf{a}_{1*}^T \\ \mathbf{a}_{2*}^T \\ \vdots \\ \mathbf{a}_{*n}^T \end{pmatrix}$$

Important Matrices Operations

- Multiplication by a scalar
- Sum (commutative, associative)
- Multiplication by a vector
- Product (not commutative)
- Inversion (square, full rank)
- Transposition

Scalar Multiplication & Sum

- In the scalar multiplication, every element of the vector or matrix is multiplied with the scalar
- The sum of two vectors is a vector consisting of the pair-wise sums of the individual entries
- The sum of two matrices is a matrix consisting of the pair-wise sums of the individual entries

Matrix Vector Product

- The i^{th} component of $\mathbf{A} \cdot \mathbf{b}$ is the dot product
  . $\mathbf{a}_{i*}^T \cdot \mathbf{b}$
- The vector $\mathbf{A} \cdot \mathbf{b}$ is linearly dependent from $\{\mathbf{a}_{*i}\}$ with coefficients $\{b_i\}$

$$\mathbf{A} \cdot \mathbf{b} = \begin{pmatrix} \mathbf{a}_{1*}^T \\ \mathbf{a}_{2*}^T \\ \vdots \\ \mathbf{a}_{n*}^T \end{pmatrix} \cdot \mathbf{b} = \begin{pmatrix} \mathbf{a}_{1*}^T \cdot \mathbf{b} \\ \mathbf{a}_{2*}^T \cdot \mathbf{b} \\ \vdots \\ \mathbf{a}_{n*}^T \cdot \mathbf{b} \end{pmatrix} = \sum_k \mathbf{a}_{*k} \cdot b_k$$

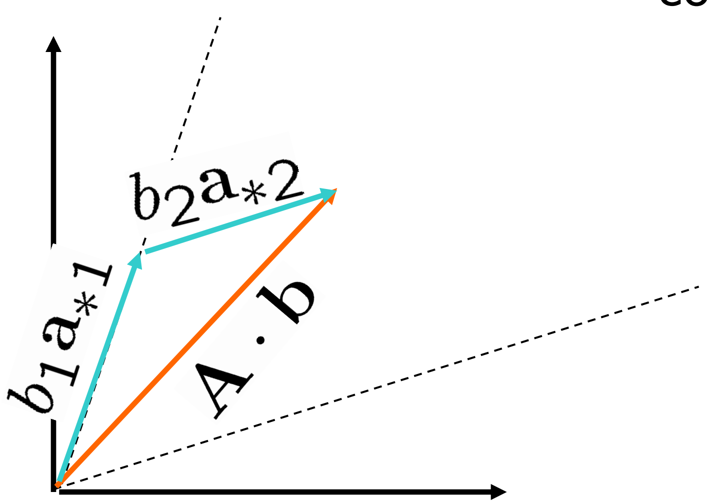
row vectors

column vectors

Matrix Vector Product

- If the column vectors of $\mathbf{A}$ represent a reference system, the product $\mathbf{A} \cdot \mathbf{b}$ computes the global transformation of the vector $\mathbf{b}$ according to $\{\mathbf{a}_{*i}\}$

column vectors



Matrix Matrix Product

- Can be defined through
  - the dot product of row and column vectors
  - the linear combination of the columns of ***A*** scaled by the coefficients of the columns of ***B***

$$
\begin{aligned}
\mathbf{C} &= \mathbf{A}\mathbf{B} \\
&= \begin{pmatrix}
\mathbf{a}_{1*}^T \cdot \mathbf{b}_{*1} & \mathbf{a}_{1*}^T \cdot \mathbf{b}_{*2} & \cdots & \mathbf{a}_{1*}^T \cdot \mathbf{b}_{*m} \\
\mathbf{a}_{2*}^T \cdot \mathbf{b}_{*1} & \mathbf{a}_{2*}^T \cdot \mathbf{b}_{*2} & \cdots & \mathbf{a}_{2*}^T \cdot \mathbf{b}_{*m} \\
\vdots & & & \\
\mathbf{a}_{n*}^T \cdot \mathbf{b}_{*1} & \mathbf{a}_{n*}^T \cdot \mathbf{b}_{*2} & \cdots & \mathbf{a}_{n*}^T \cdot \mathbf{b}_{*m}
\end{pmatrix} \\
&= \begin{pmatrix} \mathbf{A} \cdot \mathbf{b}_{*1} & \mathbf{A} \cdot \mathbf{b}_{*2} & \ldots \mathbf{A} \cdot \mathbf{b}_{*m} \end{pmatrix}
\end{aligned}
$$

Matrix Matrix Product

- If we consider the second interpretation, we see that the columns of ***C*** are the "global transformations" of the columns of ***B*** through ***A***
- All the interpretations made for the matrix vector product hold

$$
\begin{aligned}
\mathbf{C} &= \mathbf{A}\mathbf{B} \\
&= \left(\mathbf{A} \cdot \mathbf{b}_{*1} \quad \mathbf{A} \cdot \mathbf{b}_{*2} \quad \ldots \mathbf{A} \cdot \mathbf{b}_{*m} \right) \\
\mathbf{c}_{*i} &= \mathbf{A} \cdot \mathbf{b}_{*i}
\end{aligned}
$$

Linear Systems (1)

$$\mathrm{A}\mathrm{x} = \mathrm{b}$$

Interpretations:

- A set of linear equations
- A way to find the coordinates ***x*** in the reference system of ***A*** such that ***b*** is the result of the transformation of ***Ax***
- Solvable by Gaussian elimination (as taught in school)

Linear Systems (2)

$$\mathrm{Ax} = \mathrm{b}$$

Notes:

- Many efficient solvers exit, e.g., conjugate gradients, sparse Cholesky decomposition
- One can obtain a reduced system ***(A′, b′)*** by considering the matrix ***(A, b)*** and suppressing all the rows which are linearly dependent
- Let ***A'x=b'*** the reduced system with ***A':****n'xm* and ***b'****:n'x1* and rank **A'** = *min(n',m)*   rows   columns
- The system might be either over-constrained *(n′>m)* or under-constrained *(n′<m)*

Over-Constrained Systems

- "More (indep) equations than variables"
- An over-constrained system does not admit an **exact solution**
- However, if *rank* ***A'*** = *cols(**A**)* one may find a **minimum norm solution** by closed form pseudo inversion

$$\mathbf{x} = \underset{\mathbf{x}}{\operatorname{argmin}} ||\mathbf{A}'\mathbf{x} - \mathbf{b}'|| = (\mathbf{A}'^T\mathbf{A}')^{-1}\mathbf{A}'^T\mathbf{b}'$$

Note: rank = Maximum number of linearly independent rows/columns

Under-Constrained Systems

- “More variables than (indep) equations”
- The system is **under-constrained** if the number of linearly independent rows (or columns) of $\boldsymbol{A'}$ is smaller than the dimension of $\boldsymbol{b'}$
- An under-constrained system admits infinite solutions
- The degree of these infinite solutions is $cols(\boldsymbol{A'}) - rows(\boldsymbol{A'})$

Inverse

$$\mathrm{AB} = \mathrm{I}$$

- If A is a square matrix of full rank, then there is a unique matrix $\boldsymbol{B}=\boldsymbol{A}^{-1}$ such that $\boldsymbol{AB}=\boldsymbol{I}$ holds
- The i^{th} row of $\boldsymbol{A}$ is and the j^{th} column of $\boldsymbol{A}^{-1}$ are:
  - orthogonal (if $i \neq j$)
  - or their dot product is 1 (if $i = j$)

Matrix Inversion

$$\mathbf{AB} = \mathbf{I}$$

- The i^{th} column of $\boldsymbol{A}^{-1}$ can be found by solving the following linear system:

$$\mathbf{A}{\mathbf{a}^{-1}}_{*i} = \mathbf{i}_{*i}$$

This is the i^{th} column of the identity matrix

Trace (tr)

- Only defined for **square matrices**
- **Sum** of the elements on the main diagonal, that is

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

- It is a linear operator with the following properties
  - Additivity: $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
  - Homogeneity: $\mathrm{tr}(cA) = c \times \mathrm{tr}(A)$
  - Pairwise commutative: $\mathrm{tr}(AB) = \mathrm{tr}(BA), \quad \mathrm{tr}(ABC) \neq \mathrm{tr}(ACB)$

- Trace is similarity invariant $\mathrm{tr}(P^{-1}AP) = \mathrm{tr}((AP^{-1})P) = \mathrm{tr}(A)$

- Trace is transpose invariant $\mathrm{tr}(A) = \mathrm{tr}(A^T)$

- Given two vectors ***a*** and ***b***, $tr(\boldsymbol{a}^T \boldsymbol{b}) = tr(\boldsymbol{a}\,\boldsymbol{b}^T)$

Rank

- **Maximum** number of linearly independent rows (columns)
- Dimension of the **image** of the transformation $f(\mathbf{x}) = A\mathbf{x}$

- When A is $m \times n$ we have
  - $\mathrm{rank}(A) \geq 0$ and the equality holds iff A is the null matrix
  - $\mathrm{rank}(A) \leq \min(m, n)$
  - $f(\mathbf{x})$ is **injective** iff $\mathrm{rank}(A) = n$
  - $f(\mathbf{x})$ is **surjective** iff $\mathrm{rank}(A) = m$
  - if $m = n$, $f(\mathbf{x})$ is **bijective** and A is **invertible** iff $\mathrm{rank}(A) = n$

- Computation of the rank is done by
  - Gaussian elimination on the matrix
  - Counting the number of non-zero rows

Determinant (det)

- Only defined for **square matrices**
- The inverse of $\mathbf{A}$ exists if and only if $det(\mathbf{A}) \neq 0$
- For 2×2 matrices:

  Let $\mathbf{A} = [a_{ij}]$ and $|\mathbf{A}| = det(\mathbf{A})$, then

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

- For 3×3 matrices the Sarrus rule holds:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$-a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{11}$$

Determinant

- For **general** $n \times n$ matrices?

  Let $\mathbf{A}_{ij}$ be the submatrix obtained from $\mathbf{A}$ by deleting the $i\text{-}th$ row and the $j\text{-}th$ column

$$\begin{bmatrix} 1 & 2 & 5 & 0 \\ 2 & 3 & 4 & -1 \\ -5 & 8 & 0 & 0 \\ 0 & 4 & -2 & 0 \end{bmatrix} \Rightarrow \mathbf{A}_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

  Rewrite determinant for 3×3 matrices:

$$\begin{aligned} det(\mathbf{A}^{3\times3}) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{11} \\ &= a_{11} \cdot det(\mathbf{A}_{11}) - a_{12} \cdot det(\mathbf{A}_{12}) + a_{13} \cdot det(\mathbf{A}_{13}) \end{aligned}$$

Determinant

- For **general** $n \times n$ matrices?

$$\begin{aligned} det(\mathbf{A}) &= a_{11}det(\mathbf{A}_{11}) - a_{12}det(\mathbf{A}_{12}) + ... + (-1)^{1+n}a_{1n}det(\mathbf{A}_{1n}) \\ &= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}det(\mathbf{A}_{1j}) \end{aligned}$$

Let $\mathbf{C}_{ij} = (-1)^{i+j}det(\mathbf{A}_{ij})$ be the *(i,j)*-cofactor, then

$$\begin{aligned} det(\mathbf{A}) &= a_{11}\mathbf{C}_{11} + a_{12}\mathbf{C}_{12} + ... + a_{1n}\mathbf{C}_{1n} \\ &= \sum_{j=1}^{n} a_{1j}\mathbf{C}_{1j} \end{aligned}$$

This is called the **cofactor expansion** across the first row

Determinant

- **Problem:** Take a 25 x 25 matrix (which is considered small). The cofactor expansion method requires n! multiplications. For n = 25, this is 1.5 x 10^25 multiplications for which a today supercomputer would take **500,000 years**.

- There are **much faster methods**, namely using **Gauss elimination** to bring the matrix into triangular form.

$$\mathbf{A} = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & d_3 & * \\ 0 & 0 & 0 & d_4 \end{bmatrix} \qquad det(\mathbf{A}) = \prod_{i=1}^{n} d_i$$

Because for **triangular matrices** the determinant is the product of diagonal elements

Determinant: Properties

- **Row operations** ($\mathbf{A}$ is still a $n \times n$ square matrix)
  - If $\mathbf{B}$ results from $\mathbf{A}$ by interchanging two rows, then $det(\mathbf{B}) = -det(\mathbf{A})$
  - If $\mathbf{B}$ results from $\mathbf{A}$ by multiplying one row with a number c, then $det(\mathbf{B}) = c \cdot det(\mathbf{A})$
  - If $\mathbf{B}$ results from $\mathbf{A}$ by adding a multiple of one row to another row, then $det(\mathbf{B}) = det(\mathbf{A})$
- **Transpose**: $det(\mathbf{A}^T) = det(\mathbf{A})$
- **Multiplication**: $det(\mathbf{A} \cdot \mathbf{B}) = det(\mathbf{A}) \cdot det(\mathbf{B})$
- Does **not** apply to addition! $det(\mathbf{A} + \mathbf{B}) \neq det(\mathbf{A}) + det(\mathbf{B})$

Determinant: Applications

- Find **the inverse** $\mathbf{A}^{-1}$ using Cramer's rule $\mathbf{A}^{-1} = \dfrac{\mathrm{adj}(\mathbf{A})}{det(\mathbf{A})}$ with $\mathrm{adj}(\mathbf{A})$ being the adjugate of $\mathbf{A}$

$$adj(\mathbf{A}) = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

with $\boldsymbol{C_{ij}}$ being the cofactors of $\boldsymbol{A}$, i.e.,

$$\mathbf{C}_{ij} = (-1)^{i+j} det(\mathbf{A}_{ij})$$

Determinant: Applications

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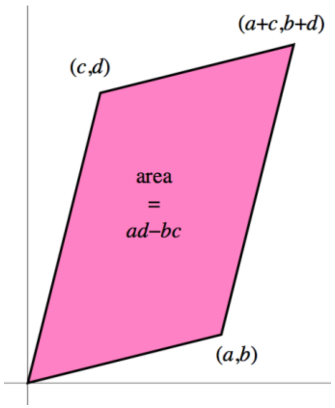
- Compute **Eigenvalues:**
  Solve the characteristic polynomial $det(\mathbf{A} - \lambda \cdot \mathbf{I}) = 0$

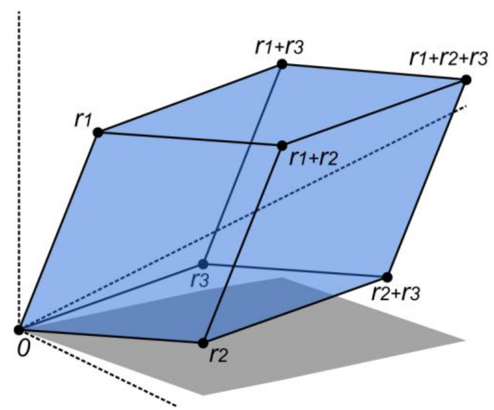
- **Area** and **Volume:** $\mathrm{area} = |det(\mathbf{A})|$

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

(r_i is *i-th* row)



Orthonormal Matrix

- A matrix Q is **orthonormal** iff its column (row) vectors represent an **orthonormal** basis

$$q_{*i}^T \cdot q_{*j} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} , \forall\ i, j$$

- As linear transformation, it is **norm** preserving

- Some properties:
  - The transpose is the inverse $QQ^T = Q^TQ = I$
  - Determinant has unity norm (§ 1)

$$1 = det(I) = det(Q^TQ) = det(Q)det(Q^T) = det(Q)^2$$

Rotation Matrix

- A Rotation matrix is an orthonormal matrix with det =+1
  - 2D Rotations $$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$
  - 3D Rotations along the main axes

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

- **IMPORTANT: Rotations are not commutative**

$$R_x(\frac{\pi}{4})\cdot R_y(\frac{\pi}{4}) = \begin{bmatrix} 0.707 & 0 & -0.707 \\ -0.5 & 0.707 & -0.5 \\ 0.5 & 0.707 & 0.5 \end{bmatrix}, \; R_x(\frac{\pi}{4})\cdot R_y(\frac{\pi}{4})\cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.414 \\ 0.586 \\ 3.414 \end{bmatrix}$$

$$R_y(\frac{\pi}{4})\cdot R_x(\frac{\pi}{4}) = \begin{bmatrix} 0.707 & -0.5 & -0.5 \\ 0 & 0.707 & -0.707 \\ 0.707 & 0.5 & 0.5 \end{bmatrix}, \; R_y(\frac{\pi}{4})\cdot R_x(\frac{\pi}{4})\cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1.793 \\ 0.707 \\ 3.207 \end{bmatrix}$$

Matrices to Represent Affine Transformations

- A general and easy way to describe a 3D transformation is via matrices

Translation Vector

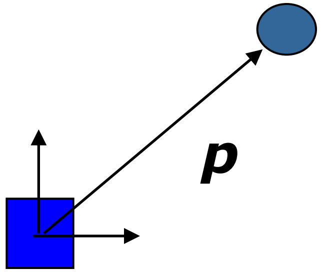
$$\mathbf{A} = \begin{pmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \quad \mathbf{A}^{-1} = \begin{pmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{t} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \quad \mathbf{p} = \begin{pmatrix} \mathbf{t} \\ \mathbf{1} \end{pmatrix}$$

Rotation Matrix

- Takes naturally into account the non-commutativity of the transformations
- See: homogeneous coordinates

Combining Transformations

- A simple interpretation: chaining of transformations (represented as homogeneous matrices)
  - Matrix ***A*** represents the pose of a **robot** in the space
  - Matrix ***B*** represents the position of a sensor on the robot
  - The **sensor** perceives an **object** at a given location ***p***, in its own frame [the sensor has no clue on where it is in the world]
  - Where is the object in the global frame?



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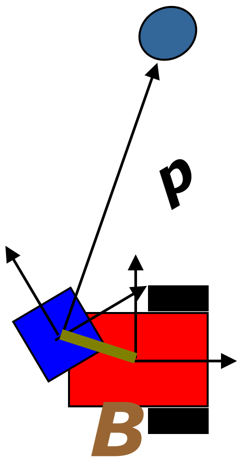

Bp gives the pose of the object wrt the robot

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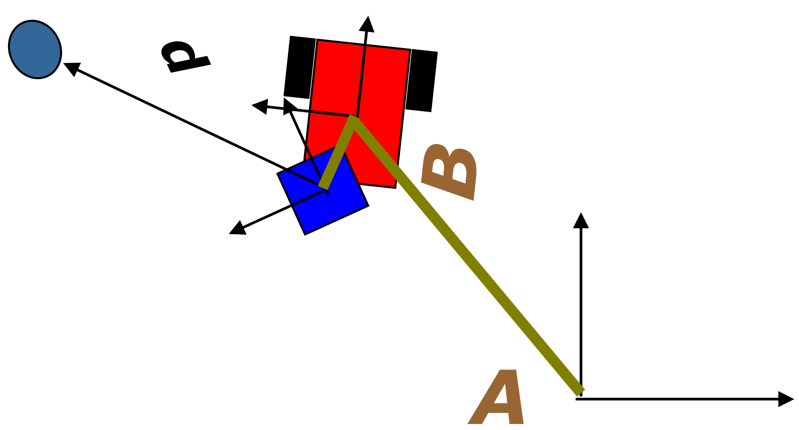


Bp gives the pose of the object wrt the robot

ABp gives the pose of the object wrt the world

Symmetric Matrix

- A matrix A is **symmetric** if $A = A^T$, e.g. $\begin{bmatrix} 1 & 4 & -2 \\ 4 & -1 & 3 \\ -2 & 3 & 5 \end{bmatrix}$

- A matrix A is **skew-symmetric** if $A = -A^T$, e.g. $\begin{bmatrix} 0 & 4 & -2 \\ -4 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

- **Every** symmetric matrix:
  - is **diagonalizable** $D = QAQ^T$, where D is a diagonal matrix of **eigenvalues** and Q is an orthogonal matrix whose columns are the **eigenvectors** of A
  - define a **quadratic form** $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \sum_{i,,j=1}^{n} a_{ij}x_i x_j$

Positive Definite Matrix

- The analogous of positive number
- Definition $M > 0$ iff $z^T M z > 0 \forall z \neq 0$
- Example
  - $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_1^2 + z_2^2 > 0$

Positive Definite Matrix

- Properties
  - **Invertible**, with positive definite inverse
  - All real **eigenvalues** > 0
  - **Trace** is > 0
  - **Cholesky** decomposition $A = LL^T$

Jacobian Matrix

- It is a **non-square matrix** $n \times m$ in general
- Given a vector-valued function

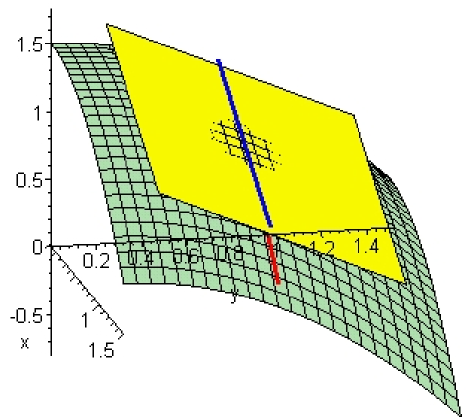
$$f(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{bmatrix}$$

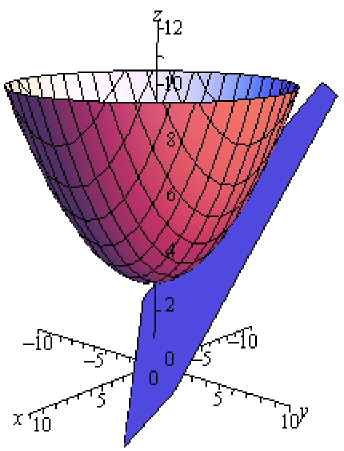
- Then, the **Jacobian matrix** is defined as

$$\mathbf{F_x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Jacobian Matrix

- It is the orientation of the **tangent plane** to the vector-valued function at a given point





- **Generalizes the gradient** of a scalar valued function

Quadratic Forms

- Many functions can be locally approximated with **quadratic form**

$$\begin{aligned} f(\mathbf{x}) &= \sum_{i,j} a_{ij} x_i x_j + \sum_i b_i x_i + c \\ &= \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b} \mathbf{x} + c \end{aligned}$$

- Often, one is interested in finding the minimum (or maximum) of a quadratic form, i.e.,

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \, f(\mathbf{x})$$

Quadratic Forms

- Question: How to efficiently compute a solution to this minimization problem

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\text{argmin}}\, f(\mathbf{x})$$

- At the minimum, we have $f'(\hat{\mathbf{x}}) = \mathbf{0}$
- By using the definition of matrix product, we can compute *f'*

$$\begin{aligned} f(\mathbf{x}) &= \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{x} + c \\ f'(\mathbf{x}) &= \mathbf{A}^T\mathbf{x} + \mathbf{A}\mathbf{x} + \mathbf{b} \end{aligned}$$

Quadratic Forms

- The minimum of $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{x} + c$ is where its derivative is 0

$$0 = \mathbf{A}^T\mathbf{x} + \mathbf{A}\mathbf{x} + \mathbf{b}$$

- Thus, we can solve the system

$$(\mathbf{A}^T + \mathbf{A})\mathbf{x} = -\mathbf{b}$$

- If the matrix is symmetric, the system becomes

$$2\mathbf{A}\mathbf{x} = -\mathbf{b}$$

- Solving that, leads to the minimum

Further Reading

- A “quick and dirty” guide to matrices is the Matrix Cookbook available at:

  http://matrixcookbook.com