Foundations of AI
3. Solving Problems by Searching

Problem-Solving Agents, Formulating Problems, Search Strategies
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Problem-Solving Agents

→ Goal-based agents
Formulation: goal and problem
Given: initial state
Goal: To reach the specified goal (a state) through the execution of appropriate actions.

→ Search for a suitable action sequence and execute the actions

A Simple Problem-Solving Agent

function SIMPLE-PROBLEM-SOLVING-AGENT(percept) returns an action
inputs: percept, a percept
static: seq, an action sequence, initially empty
        state, some description of the current world state
        goal, a goal, initially null
        problem, a problem formulation

state ← UPDATE-STATE(state, percept)
if seq is empty then do
    goal ← FORMULATE-GOAL(state)
    problem ← FORMULATE-PROBLEM(state, goal)
    seq ← SEARCH(problem)
end if
action ← FIRST(seq)
seq ← REST(seq)
return action
Properties of this Agent

- static world
- observable environment
- discrete states
- deterministic environment

Problem Formulation

- Goal formulation
  World states with certain properties
- Definition of the state space
  (important: only the relevant aspects → abstraction
- Definition of the actions that can change the world state
- Definition of the problem type, which is dependent on the knowledge of the world states and actions → states in the search space
- Determination of the search cost (search costs, offline costs) and the execution costs (path costs, online costs)

**Note:** The type of problem formulation can have a big influence on the difficulty of finding a solution.

Example Problem Formulation

Given an n x n board from which two diagonally opposite corners have been removed (here 8x8):

Goal: Cover the board completely with dominoes, each of which covers two neighbouring squares.

→ Goal, state space, actions, search, ...

Alternative Problem Formulation

**Question:**
Can a chess board consisting of $n^2/2$ black and $n^2/2-2$ white squares be completely covered with dominoes such that each domino covers one black and one white square?

... clearly not.
Problem Formulation for the Vacuum Cleaner World

- **World state space:**
  2 positions, dirt or no dirt  
  → 8 world states

- **Actions:**
  Left (L), Right (R), or Suck (S)

- **Goal:**
  no dirt in the rooms

- **Path costs:**
  one unit per action

The Vacuum Cleaner Problem as a One-State Problem

If the environment is completely accessible, the vacuum cleaner always knows where it is and where the dirt is. The solution then is reduced to searching for a path from the initial state to the goal state.

The Vacuum Cleaner World as a Multiple-State Problem

If the vacuum cleaner has no sensors, it does not know where it is or the dirt is.

In spite of this, it can still solve the problem. Here, states are knowledge states.

States for the search: The world states 1-8.
Concepts (1)

Initial State
The state from which the agent infers that it is at the beginning

State Space
Set of all possible states

Actions
Description of possible actions and their outcome (successor function)

Goal Test
Tests whether the state description matches a goal state

Concepts (2)

Path
A sequence of actions leading from one state to another.

Path Costs
Cost function $g$ over paths. Usually the sum of the costs of the actions along the path.

Solution
Path from an initial to a goal state

Search Costs
Time and storage requirements to find a solution

Total Costs
Search costs + path costs

Example: The 8-Puzzle

- **States**: Description of the location of each of the eight tiles and (for efficiency) the blank square.
- **Initial State**: Initial configuration of the puzzle.
- **Actions or Successor function**: Moving the blank left, right, up, or down.
- **Goal Test**: Does the state match the configuration on the right (or any other configuration)?
- **Path Costs**: Each step costs 1 unit (path costs correspond to its length).
Example: 8-Queens Problem

Almost a solution:

- **States:** Any arrangement of 0 to 8 queens on the board.
- **Initial state:** No queen on the board.
- **Successor function:** Add a queen to an empty field on the board.
- **Goal test:** 8 queens on the board such that no queen attacks another
- **Path costs:** 0 (we are only interested in the solution).

Example: 8-Queens Problem

A solution:

- **States:** Any arrangement of 0 to 8 queens on the board.
- **Initial state:** No queen on the board.
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- **Goal test:** 8 queens on the board such that no queen attacks another
- **Path costs:** 0 (we are only interested in the solution).

Alternative Formulations

- **Naïve formulation**
  - **States:** Any arrangement of 0-8 queens
  - **Problem:** $64 \cdot 63 \cdot \ldots \cdot 57 \approx 10^{14}$ possible states

- **Better formulation**
  - **States:** Any arrangement of $n$ queens ($0 \leq n \leq 8$) one per column in the leftmost $n$ columns such that no queen attacks another.
  - **Successor function:** Add a queen to any square in the leftmost empty column such that it is not attacked by any other queen.
  - **Problem:** 2,057 states

  Sometimes no admissible states can be found.

Example: Missionaries and Cannibals

Informal problem description:

- Three missionaries and three cannibals are on one side of a river that they wish to cross.
- A boat is available that can hold at most two people.
- You must never leave a group of missionaries outnumbered by cannibals on the same bank.

→ Find an action sequence that brings everyone safely to the opposite bank.
**Formalization of the M&C Problem**

*States:* triple \((x,y,z)\) with \(0 \leq x,y,z \leq 3\), where \(x,y\), and \(z\) represent the number of missionaries, cannibals and boats currently on the original bank.

*Initial State:* \((3,3,1)\)

*Successor function:* From each state, either bring one missionary, one cannibal, two missionaries, two cannibals, or one of each type to the other bank.

*Note:* Not all states are attainable (e.g. \((0,0,1)\)), and some are illegal.

*Goal State:* \((0,0,0)\)

*Path Costs:* 1 unit per crossing

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**Examples of Real-World Problems**

- **Route Planning, Shortest Path Problem**
  - Simple in principle (polynomial problem). Complications arise when path costs are unknown or vary dynamically (e.g. Route planning in Canada)

- **Travelling Salesperson Problem (TSP)**
  - A common prototype for NP-complete problems

- **VLSI Layout**
  - Another NP-complete problem

- **Robot Navigation (with a high degree of freedom)**
  - Difficulty increases quickly with the level of freedom. Further possible complications: errors of perception, unknown environments

- **Assembly Sequencing**
  - Planning of the assembly of complex objects (by robots)

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**General Search**

From the initial state, produce all successive states step by step → search tree.

(a) initial state \((3,3,1)\)

(b) after expansion of \((3,3,1)\)

\[(2,3,0) (3,2,0) (2,2,0) (1,3,0) (3,1,0)\]

(c) after expansion of \((3,2,0)\)

\[(2,2,0) (1,3,0) (3,1,0)\]

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**Implementing the Search Tree**

*Data structure for nodes in the search tree:*

*State:* state in the state space

*Parent-Node:* Predecessor nodes

*Action:* The operator that generated the node

*Depth:* number of steps along the path from the initial state

*Path Cost:* Cost of the path from the initial state to the node

*Operations on a queue:*

- Make-Queue(Elements): Creates a queue
- Empty?(Queue): Empty test
- First(Queue): Returns the first element of the queue
- Remove-First(Queue): Returns the first element
- Insert(Element, Queue): Inserts new elements into the queue (various possibilities)
- Insert-All(Elements, Queue): Inserts a set of elements into the queue
Criteria for Search Strategies

Completeness:
Is the strategy guaranteed to find a solution when there is one?

Time Complexity:
How long does it take to find a solution?

Space Complexity:
How much memory does the search require?

Optimality:
Does the strategy find the best solution (with the lowest path cost)?

Search Strategies

Uninformed or blind searches:
No information on the length or cost of a path to the solution.

• breadth-first search, uniform cost search, depth-first search,
• depth-limited search, Iterative deepening search, and
• bi-directional search.

In contrast: informed or heuristic approaches
**Breadth-First Search (1)**

Nodes are expanded in the order they were produced. \( \text{fringe} = \text{FIFO-QUEUE}() \).

- Always finds the shallowest goal state first.
- Completeness.
- The solution is optimal, provided the path cost is a non-decreasing function of the depth of the node (e.g. when every action has identical, non-negative costs).

**Breadth-First Search (2)**

The costs, however, are very high. Let \( b \) be the maximal branching factor and \( d \) the depth of a solution path. Then the maximal number of nodes expanded is

\[
b + b^2 + b^3 + \ldots + b^d + (b^{d+1} - b) \in \mathcal{O}(b^{d+1})
\]

Example: \( b = 10, 10,000 \) nodes/second, \( 1,000 \) bytes/node:

<table>
<thead>
<tr>
<th>Depth</th>
<th>Nodes</th>
<th>Time</th>
<th>Memory</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1,100</td>
<td>.11 seconds</td>
<td>1 megabyte</td>
</tr>
<tr>
<td>4</td>
<td>111,100</td>
<td>11 seconds</td>
<td>106 megabytes</td>
</tr>
<tr>
<td>6</td>
<td>( 10^7 )</td>
<td>19 minutes</td>
<td>10 gigabytes</td>
</tr>
<tr>
<td>8</td>
<td>( 10^9 )</td>
<td>31 hours</td>
<td>1 terabyte</td>
</tr>
<tr>
<td>10</td>
<td>( 10^{11} )</td>
<td>129 days</td>
<td>101 terabytes</td>
</tr>
<tr>
<td>12</td>
<td>( 10^{13} )</td>
<td>35 years</td>
<td>10 petabytes</td>
</tr>
<tr>
<td>14</td>
<td>( 10^{15} )</td>
<td>3,523 years</td>
<td>1 exabyte</td>
</tr>
</tbody>
</table>

**Uniform Cost Search**

Modification of breadth-first search to always expand the node with the lowest-cost \( g(n) \).

Always finds the cheapest solution, given that \( g(\text{successor}(n)) \geq g(n) \) for all \( n \).

**Depth-First Search**

Always expands and unexpanded node at the greatest depth (Queue-Fn = Enqueue-at-front).

Example (Nodes at depth 3 are assumed to have no successors):
**Depth-Limited Search**

Depth-first search with an imposed cutoff on the maximum depth of a path. E.g. route planning: with \( n \) cities, the maximum depth is \( n-1 \).

Here, a depth of 9 is sufficient (diameter of the problem).

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**Iterative Deepening Search (1)**

- Combines depth- and breadth-first searches
- Optimal and complete like breadth-first search, but requires less memory

```plaintext
function ITERATIVE-DEEPENING-SEARCH(problem) returns a solution sequence
    inputs: problem, a problem
    for depth ← 0 to ∞ do
        if DEPTH-LIMITED-SEARCH(problem, depth) succeeds then return its result
    end
    return failure
```

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**Example**

- Limit = 0
- Limit = 1
- Limit = 2
- Limit = 3

---

**Iterative Deepening Search (2)**

Number of expansions

<table>
<thead>
<tr>
<th>Iterative Deepening Search</th>
<th>((d^2 + (d-1)b^2 + \ldots + 3b^2 + 2b + 1)b + 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Breadth-First-Search</td>
<td>(b + b^2 + \ldots + b^d + b^d + b^d + b^d)</td>
</tr>
</tbody>
</table>

Example: \( b = 10, d = 5 \)

| Breadth-First-Search       | \(10 + 100 + 1,000 + 10,000 + 999,990\) |
|----------------------------|\(= 1,111,100\) |
| Iterative Deepening Search | \(50 + 400 + 3,000 + 20,000 + 100,000\) |
|                            |\(= 123,450\) |

For \( b = 10 \), only 11% of the nodes expanded by breadth-first-search are generated, so that the memory requirement is considerably lower.

Time complexity: \(O(b^d)\) Memory complexity: \(O(b\cdot d)\)

→ Iterative deepening in general is the preferred uninformed search method when there is a large search space and the depth of the solution is not known.
Bidirectional Searches

As long as forwards and backwards searches are symmetrical, search times of \( O(2 \cdot b^{d/2}) = O(b^{d/2}) \) can be reached.

e.g. for \( b=10, d=6 \), instead of 111111 only 2222 nodes!

Problems with Bidirectional Search

- The operators are not always reversible, which makes calculation the predecessors very difficult.
- In some cases there are many possible goal states, which may not be easily describable. Example: The predecessors of the checkmate in chess.
- There must be an efficient way to check if a new node already appears in the search tree of the other half of the search.
- What kind of search should be chosen for each direction (the previous figure shows a breadth-first search, which is not always optimal)?

Comparison of Search Strategies

Time complexity, space complexity, optimality, completeness

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Breadth-First</th>
<th>Uniform-Cost</th>
<th>Depth-First</th>
<th>Depth-Limited</th>
<th>Iterative Deepening</th>
<th>Bidirectional (if applicable)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complete?</td>
<td>Yes(^a)</td>
<td>Yes(^a, b)</td>
<td>No</td>
<td>No</td>
<td>Yes(^c)</td>
<td>Yes(^a, d)</td>
</tr>
<tr>
<td>Time</td>
<td>( O(b^{d+1}) )</td>
<td>( O(b^{d^2/c}) )</td>
<td>( O(b^m) )</td>
<td>( O(b^l) )</td>
<td>( O(b^l) )</td>
<td>( O(b^{l/2}) )</td>
</tr>
<tr>
<td>Space</td>
<td>( O(b^{l+1}) )</td>
<td>( O(b^{l^2/c}) )</td>
<td>( O(bm) )</td>
<td>( O(bl) )</td>
<td>( O(bld) )</td>
<td>( O(b^{l/2}) )</td>
</tr>
<tr>
<td>Optimal?</td>
<td>Yes(^c)</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>Yes(^c)</td>
<td>Yes(^a, d)</td>
</tr>
</tbody>
</table>

The table above provides a comparison of search strategies based on time complexity, space complexity, optimality, and completeness. The criteria listed include completeness, time, space, and optimality, each with specific complexities and optimality attributes. The table includes variables such as branching factor, depth of solution, maximum depth of the search tree, depth limit, and cost of the optimal solution, Minimal cost of an action. Superscripts include:

- \(^a\) b is finite
- \(^b\) if step costs not less than \( \varepsilon \)
- \(^c\) if step costs are all identical
- \(^d\) if both directions use breadth-first search

Constraint Satisfaction Problems (CSPs)

CSPs constitute a special class of problems that, in light of structural restrictions, permit special search techniques.

- States are defined by the values of variables.
- Actions assign a value to a variable.
- The goal test is defined through constraints that the variable assignments must satisfy.
- A goal state is an assignment of variables with values that satisfy all constraints.
Example: the 8-Queens Problem as a CSP

- There are 8 variables, \( V_1, \ldots, V_8 \), where \( V_i \) represents the queen in the \( i \)th column.
- \( V_i \) can take on a value from \( \{1, 2, \ldots, 8\} \) representing the goal position.
- Constraints exist between each pair of variables to express the non-attackability factor.
- \( \rightarrow \) Discrete, finite, binary CSP

Graph Coloring as a CSP

Given a graph with \( n \) nodes, colour the nodes with \( k \) colours such that no two nodes connected by an edge have the same colour. (\( \rightarrow \) NP-complete problem!)

Example:

can the following graph be coloured with 3 colours?

General Search Applied to CSPs

- Depth-first search for CSPs is complete, as a maximum of only \( n \) operators (assignments of values to variables) are possible.
- Naive implementations have a very high branching factor! Let \( D_i \) be the set of possible values for \( V_i \). Then the branching factor is \( b = \sum_{i=1}^{n} |D_i| \).
- The order of instantiation of the variables is irrelevant to the solution. Therefore one variable can be chosen (non-deterministically) at each expansion; the branching factor is \( (\sum_{i=1}^{n} |D_i|)/n \).
- Whenever an action has been carried out, check if any constraints are violated. If so, it is unnecessary to expand the current node.
- \( \rightarrow \) Depth-first search + Test = Backtracking.

Forward Checking and Arc Consistency

Suppose during backtracking a decision is made that renders a solution impossible. This can only be detected in the leaves of the tree.
- \( \rightarrow \) unnecessary exploration of the subtree.

Solutions:
- Forward Checking: Delete from the domains of the as-yet-uninstantiated variables the values that conflict with the variables assigned so far. Backtrack when the domain of a variable becomes empty.
- Special form of forward checking: Arc Consistency. A state is arc consistent if every variable has a value in its domain that is consistent with each of the constraints on that variable.
Example

Instantiation $V_1 = \text{red}$.

Creation of arc consistency.

Instantiation: $V_3 = \text{green}$ and creation of arc consistency.

Summary

- Before an agent can start searching for solutions, it must formulate a goal and then use that goal to formulate a problem.

- A problem consists of five parts: The state space, initial situation, actions, goal test, and path costs. A path from an initial state to a goal state is a solution.

- A general search algorithm can be used to solve any problem. Specific variants of the algorithm can use different search strategies.

- Search algorithms are judged on the basis of completeness, optimality, time complexity, and space complexity.

- Constraint satisfaction problems constitute a special class of problems that permit special search techniques.