5. Constraint Satisfaction Problems

CSPs as Search Problems, Solving CSPs, Problem Structure

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- What are CSPs?
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Constraint Satisfaction Problems

- In search problems, the state does not have a structure (everything is in the data structure) – in CSPs states are explicitly represented as variable assignments.
- A CSP consists of
  - a set of variables \( \{x_1, x_2, \ldots, x_n\} \) to which
  - values \( \{d_1, d_2, \ldots, d_k\} \) can be assigned
  - respecting a set of constraints over the variables
- A CSP is solved by a variable assignment that satisfies all given constraints
- Formal representation language with associated general inference algorithms
Example: Map-Coloring

- **Variables:** WA, NT, SA, Q, NSW, V, T
- **Values:** \{red, green, blue\}
- **Constraints:** adjacent regions must have different colors, e.g., NSW \(\neq\) V
Australian Capital Territory (ACT) and Canberra (inside NSW)

View of the Australian National University and Telstra Tower
One Solution

Solution assignment:

- \{ WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green \}
- Perhaps in addition ACT = blue
Constraint Graph

- Works for **binary** CSPs (otherwise hypergraph)
- **Nodes** = variables, **arcs** = constraints
- Graph structure can be important (e.g., connected components)

**Note:** Our problem is 3-colorability for a planar graph
Variations

- Binary, ternary, or even higher arity
- **Finite** domains (d values) => \( d^n \) possible variable assignments
- **Infinite** domains (reals, integers)
  - *linear constraints* solvable (in P if real)
  - *nonlinear constraints* unsolvable
Applications

- Timetabling (classes, rooms, times)
- Configuration (hardware, cars, ...)
- Spreadsheets
- Scheduling
- Floor planning
- Frequency assignments
- ...

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Backtracking Search over Assignments

- Assign values to variables step by step (order does not matter)
- Consider only one variable per search node!
- DFS with single-variable assignments is called backtracking search
- Can solve $n$-queens for $n \approx 25$
Algorithm

function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING([], csp)

function RECURSIVE-BACKTRACKING(assigns, csp) returns solution/failure
    if assigns is complete then return assigns
    var ← SELECT-UNASSIGNED-VARIABLE(ARIABLES[csp], assigns, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assigns, csp) do
        if value is consistent with assigns according to CONSTRAINTS[csp] then
            result ← RECURSIVE-BACKTRACKING([var = value | assigns], csp)
            if result ≠ failure then return result
        end
    end
    return failure
Example (1)
Example (2)
Example (3)
Example (4)
Improving Efficiency: CSP Heuristics & Pruning Techniques

- **Variable ordering**: Which one to assign first?
- **Value ordering**: Which value to try first?
- Try to detect failures early on
- Try to exploit problem structure

- **Note**: all this is not problem-specific!
Variable Ordering: Most constrained first

- Most constrained variable:
  - choose the variable with the fewest remaining legal values
  - reduces branching factor!
Variable Ordering: Most Constraining Variable First

- Break ties among variables with the same number of remaining legal values:
  - choose variable with the most constraints on remaining unassigned variables
  - reduces branching factor in the next steps
Value Ordering: Least Constraining Value First

- Given a variable,
  - choose first a value that rules out the fewest values in the remaining unassigned variables
  - We want to find an assignment that satisfies the constraints (of course, does not help if unsat.)
Rule Out Failures Early On: Forward Checking

- Whenever a value is assigned to a variable, values that are now illegal for other variables are removed.
- Implements what the ordering heuristics implicitly compute.
- $WA = \text{red}$, then $NT$ cannot become $\text{red}$.
- If all values are removed for one variable, we can stop!
Forward Checking (1)

- Keep track of remaining values
- Stop if all have been removed
Forward Checking (2)

- Keep track of remaining values
- Stop if all have been removed
Forward Checking (3)

- Keep track of remaining values
- Stop if all have been removed
Forward Checking (4)

- Keep track of remaining values
- Stop if all have been removed
Forward Checking: Sometimes it Misses Something

- Forward Checking propagates information from assigned to unassigned variables
- However, there is no propagation between unassigned variables
Arc Consistency

- A directed arc $X \rightarrow Y$ is “consistent” iff
  - for every value $x$ of $X$, there exists a value $y$ of $Y$, such that $(x,y)$ satisfies the constraint between $X$ and $Y$
- Remove values from the domain of $X$ to enforce arc-consistency
- Arc consistency detects failures earlier
- Can be used as preprocessing technique or as a propagation step during backtracking
Arc Consistency Example
AC3 Algorithm

function AC-3(csp) returns the CSP, possibly with reduced domains
  inputs: csp, a binary CSP with variables \{X_1, X_2, \ldots, X_n\}
  local variables: queue, a queue of arcs, initially all the arcs in csp
  while queue is not empty do
    \((X_i, X_j) \leftarrow \text{Remove-First}(\text{queue})\)
    if Remove-Inconsistent-Values(\(X_i, X_j\)) then
      for each \(X_k\) in Neighbors[\(X_i\)] do
        add \((X_k, X_i)\) to queue

function Remove-Inconsistent-Values(\(X_i, X_j\)) returns true iff we remove a value
  removed \leftarrow\ false
  for each \(x\) in Domain[\(X_i\)] do
    if no value \(y\) in Domain[\(X_j\)] allows \((x, y)\) to satisfy the constraint between \(X_i\) and \(X_j\)
    then delete \(x\) from Domain[\(X_i\)]; removed \leftarrow\ true
  return removed
Properties of AC3

- AC3 runs in $O(d^3n^2)$ time, with $n$ being the number of nodes and $d$ being the maximal number of elements in a domain.

- Of course, AC3 does not detect all inconsistencies (which is an NP-hard problem).
Problem Structure (1)

- CSP has two independent components
- Identifiable as connected components of constraint graph
- Can reduce the search space dramatically
Problem Structure (2): Tree-structured CSPs

If the CSP graph is a tree, then it can be solved in $O(nd^2)$
- General CSPs need in the worst case $O(d^n)$

Idea: Pick root, order nodes, apply arc consistency from leaves to root, and assign values starting at root
Problem Structure (2): Tree-structured CSPs

- Apply arc-consistency to \((X_i, X_k)\), when \(X_i\) is the parent of \(X_k\), for all \(k=n\) \textit{downto} 2.
- Now one can start at \(X_1\) \textit{assigning values} from the remaining domains without creating any conflict in one sweep through the tree!
- Algorithm \textit{linear in} \(n\)
Problem Structure (3): Almost Tree-structured

- **Conditioning**: Instantiate a variable and prune values in neighboring variables

- **Cutset conditioning**: Instantiate (in all ways) a set of variables in order to reduce the graph to a tree (note: finding minimal cutset is NP-hard)
Another Method: Tree Decomposition (1)

- Decompose problem into a set of connected sub-problems, where two sub-problems are connected when they share a constraint
- Solve sub-problems independently and combine solutions
Another Method: Tree Decomposition (2)

- A tree decomposition must satisfy the following conditions:
  - Every variable of the original problem appears in at least one sub-problem
  - Every constraint appears in at least one sub-problem
  - If a variable appears in two sub-problems, it must appear in all sub-problems on the path between the two sub-problems
  - The connections form a tree
Another Method: Tree Decomposition (3)

- Consider sub-problems as new mega-nodes, which have values defined by the solutions to the sub-problems.
- Use technique for tree-structured CSP to find an overall solution (constraint is to have identical values for the same variable).
Tree Width

- Tree width of a tree decomposition = size of largest sub-problem minus 1
- Tree width of a graph is minimal tree width over all possible tree decompositions
- If a graph has tree width $w$ and we know a tree decomposition with that width, we can solve the problem in $O(nd^{w+1})$
- Finding a tree decomposition with minimal tree width is NP-hard
Summary & Outlook

- **CSPs** are a special kind of search problem:
  - states are value assignments
  - goal test is defined by constraints
- **Backtracking** = DFS with one variable assigned per node. Other intelligent backtracking techniques possible
- **Variable/value ordering** heuristics can help dramatically
- **Constraint propagation** prunes the search space
- **Path-consistency** is a constraint propagation technique for triples of variables
- **Tree structure** of CSP graph simplifies problem significantly
- **Cutset conditioning** and **tree decomposition** are two ways to transform part of the problem into a tree
- CSPs can also be solved using **local search**