

# Foundations of AI

## 18. Strategic Games

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*Strategic Reasoning and Acting*

*Wolfram Burgard and Bernhard Nebel*

# Strategic Game

- A **strategic game**  $G$  consists of
  - a finite set  $N$  (the set of **players**)
  - for each player  $i \in N$  a non-empty set  $A_i$  (the set of **actions** or **strategies** available to player  $i$ ), whereby  $A = \prod_i A_i$
  - for each player  $i \in N$  a function  $u_i: A \rightarrow R$  (the **utility** or **payoff** function)
  - $G = (N, (A_i), (u_i))$
- If  $A$  is finite, then we say that the game is *finite*

# Playing the Game

- Each player  $i$  makes a **decision** which action to play:  $a_i$
- All players make their moves simultaneously leading to the **action profile**  $a^* = (a_1, a_2, \dots, a_n)$
- Then each player gets the **payoff**  $u_i(a^*)$
- Of course, each player tries to maximize its own payoff, but what is the right decision?
- **Note:** While we want to maximize our payoff, we are not interested in harming our opponent. It just does not matter to us what he will get!
  - If we want to model something like this, the payoff function must be changed

# Notation

- For *2-player games*, we use a matrix, where the strategies of **player 1** are the **rows** and the strategies of **player 2** the **columns**
- The payoff for every action profile is specified as a pair  $x,y$ , whereby  $x$  is the value for player 1 and  $y$  is the value for player 2
- Example: For (T,R), **player 1** gets  $x_{12}$ , and **player 2** gets  $y_{12}$

|                     | Player 2<br>L action | Player 2<br>R action |
|---------------------|----------------------|----------------------|
| Player1<br>T action | $x_{11}, y_{11}$     | $x_{12}, y_{12}$     |
| Player1<br>B action | $x_{21}, y_{21}$     | $x_{22}, y_{22}$     |

# Example Game: Bach and Stravinsky

- Two people want to out together to a concert of music by either Bach or Stravinsky. Their main concern is to go out together, but one prefers Bach, the other Stravinsky. Will they meet?
- This game is also called the *Battle of the Sexes*

|            | Bach | Stravinsky |
|------------|------|------------|
| Bach       | 2,1  | 0,0        |
| Stravinsky | 0,0  | 1,2        |

# Example Game: Hawk-Dove

- Two animals fighting over some prey.
- Each can behave like a dove or a hawk
- The best outcome is if oneself behaves like a hawk and the opponent behaves like a dove
- This game is also called *chicken*.

|      | Dove | Hawk |
|------|------|------|
| Dove | 3,3  | 1,4  |
| Hawk | 4,1  | 0,0  |

# Example Game: Prisoner's Dilemma

- Two suspects in a crime are put into separate cells.
- If they both confess, each will be sentenced to 3 years in prison.
- If only one confesses, he will be freed.
- If neither confesses, they will both be convicted of a minor offense and will spend one year in prison.

|               | Don't confess | Confess |
|---------------|---------------|---------|
| Don't confess | 3,3           | 0,4     |
| Confess       | 4,0           | 1,1     |

# Solving a Game

- What is the right move?
- Different possible **solution concepts**
  - Elimination of strictly or weakly **dominated** strategies
  - **Maximin** strategies (for minimizing the loss in zero-sum games)
  - **Nash equilibrium**
- How difficult is it to compute a solution?
- Are there always solutions?
- Are the solutions unique?

# Strictly Dominated Strategies

- Notation:
  - Let  $a = (a_i)$  be a strategy profile
  - $a_{-i} := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$
  - $(a_{-i}, a'_i) := (a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_n)$
- Strictly dominated strategy:
  - An strategy  $a_j^* \in A_j$  is *strictly dominated* if there exists a strategy  $a'_j$  such that for all strategy profiles  $a \in A$ :
$$u_j(a_{-j}, a'_j) > u_j(a_{-j}, a_j^*)$$
- Of course, it is **not rational** to play **strictly dominated strategies**

# Iterated Elimination of Strictly Dominated Strategies

- Since strictly dominated strategies will never be played, one can **eliminate** them from the game
- This can be done **iteratively**
- If this converges to a single strategy profile, the result is **unique**
- This can be regarded as the **result** of the game, because it is the **only rational outcome**

# Iterated Elimination: Example

- Eliminate:

    , dominated by

    , dominated by

    , dominated by

    , dominated by

    , dominated by

    , dominated by

- Result:

|    | b1  | b2   | b3  | b4   |
|----|-----|------|-----|------|
| a1 | 1,7 | 2,5  | 7,2 | 0,1  |
| a2 | 5,2 | 3,3  | 5,2 | 0,1  |
| a3 | 7,0 | 2,5  | 0,4 | 0,1  |
| a4 | 0,0 | 0,-2 | 0,0 | 9,-1 |

# Iterated Elimination: Prisoner's Dilemma

- Player 1 reasons that “not confessing” is strictly dominated and eliminates this option
- Player 2 reasons that player 1 will not consider “not confessing”. So he will eliminate this option for himself as well
- So, they both confess

|               | Don't confess | Confess |
|---------------|---------------|---------|
| Don't confess | 3,3           | 0,4     |
| Confess       | 4,0           | 1,1     |

# Weakly Dominated Strategies

- Instead of strict domination, we can also go for weak domination:
  - An strategy  $a_j^* \in A_j$  is *weakly dominated* if there exists a strategy  $a_j'$  such that for all strategy profiles  $a \in A$ :

$$u_j(a_{-j}, a_j') \geq u_j(a_{-j}, a_j^*)$$

and for at least one profile  $a \in A$ :

$$u_j(a_{-j}, a_j') > u_j(a_{-j}, a_j^*).$$

# Results of Iterative Elimination of Weakly Dominated Strategies

- The result is not necessarily unique
- Example:
  - Eliminate
  
  
  
  
  
  
  
  
  
  
  - Eliminate:

|   | L   | R   |
|---|-----|-----|
| T | 2,1 | 0,0 |
| M | 2,1 | 1,1 |
| B | 0,0 | 1,1 |

# Analysis of the *Guessing 2/3 of the Average* Game

- All strategies above 67 are weakly dominated, since they will *never ever* lead to winning the prize, so they can be eliminated!
- This means, that all strategies above  
$$\frac{2}{3} \times 67$$
can be eliminated
- ... and so on
- ... until all strategies above 1 have been eliminated!
- So: The rationale strategy would be to play 1!

# Existence of Dominated Strategies

- Dominating strategies are a convincing **solution concept**
- Unfortunately, often dominated strategies do not exist
- What do we do in this case?
  - **Nash equilibrium**

|      | Dove | Hawk |
|------|------|------|
| Dove | 3,3  | 1,4  |
| Hawk | 4,1  | 0,0  |

# Nash Equilibrium

- A *Nash equilibrium* is an action profile  $a^* \in A$  with the property that for all players  $i \in N$ :

$$u_i(a^*) = u_i(a^*_{-i}, a^*_i) \geq u_i(a^*_{-i}, a_i) \quad \forall a_i \in A_i$$

- In words, it is an action profile such that there is **no incentive** for any agent **to deviate** from it
- While it is less convincing than an action profile resulting from iterative elimination of dominated strategies, it is still a **reasonable solution concept**
- If there exists a **unique solution** from **iterated elimination of strictly dominated strategies**, then it is also a **Nash equilibrium**

# Example Nash-Equilibrium: Prisoner's Dilemma

- Don't – Don't
  - not a NE
- Don't – Confess (and vice versa)
  - not a NE
- Confess – Confess
  - NE

|               | Don't confess | Confess |
|---------------|---------------|---------|
| Don't confess | 3,3           | 0,4     |
| Confess       | 4,0           | 1,1     |

# Example Nash-Equilibrium: Hawk-Dove

- Dove-Dove:
  - not a NE
- Hawk-Hawk
  - not a NE
- Dove-Hawk
  - is a NE
- Hawk-Dove
  - is, of course, another NE
- So, NEs are not necessarily unique

|      | Dove | Hawk |
|------|------|------|
| Dove | 3,3  | 1,4  |
| Hawk | 4,1  | 0,0  |

# Auctions

- An **object** is to be **assigned** to a player in the set  $\{1, \dots, n\}$  in exchange for a payment.
- Player  $i$  **valuation** of the object is  $v_i$ , and  $v_1 > v_2 > \dots > v_n$ .
- The mechanism to assign the object is a **sealed-bid auction**: the players simultaneously submit bids (non-negative real numbers)
- The object is given to the player with the lowest index among those who submit the highest bid in exchange for the payment
- The payment for a **first price** auction is the highest bid.
- What are the Nash equilibria in this case?

# Formalization

- Game  $G = (\{1, \dots, n\}, (A_i), (u_i))$
- $A_i$ : bids  $b_i \in \mathbb{R}^+$
- $u_i(b_{-i}, b_i) = v_i - b_i$  if  $i$  has won the auction, 0 otherwise
- Nobody would bid more than his valuation, because this could lead to negative utility, and we could easily achieve 0 by bidding 0.

# Nash Equilibria for First-Price Sealed-Bid Auctions

- The Nash equilibria of this game are all profiles  $b$  with:
  - $b_i \leq b_1$  for all  $i \in \{2, \dots, n\}$ 
    - No  $i$  would bid more than  $v_2$  because it could lead to negative utility
    - If a  $b_i$  (with  $< v_2$ ) is higher than  $b_1$  player 1 could increase its utility by bidding  $v_2 + \varepsilon$
    - So 1 wins in all NEs
  - $v_1 \geq b_1 \geq v_2$ 
    - Otherwise, player 1 either loses the bid (and could increase its utility by bidding more) or would have itself negative utility
  - $b_j = b_1$  for at least one  $j \in \{2, \dots, n\}$ 
    - Otherwise player 1 could have gotten the object for a lower bid

# Another Game: Matching Pennies

- Each of two people chooses either **Head** or **Tail**. If the choices differ, player 1 pays player 2 a euro; if they are the same, player 2 pays player 1 a euro.
- This is also a **zero-sum** or **strictly competitive** game
- No NE at all! What shall we do here?

|      | Head | Tail |
|------|------|------|
| Head | 1,-1 | -1,1 |
| Tail | -1,1 | 1,-1 |

# Randomizing Actions ...

- Since there does not seem to exist a rational decision, it might be best to **randomize** strategies.
- Play **Head** with probability  $p$  and **Tail** with probability  $1-p$
- Switch to **expected utilities**

|      | Head | Tail |
|------|------|------|
| Head | 1,-1 | -1,1 |
| Tail | -1,1 | 1,-1 |

# Some Notation

- Let  $G = (N, (A_i), (u_i))$  be a **strategic game**
- Then  $\Delta(A_i)$  shall be the set of probability distributions over  $A_i$  – the set of **mixed strategies**  
 $\alpha_i \in \Delta(A_i)$
- $\alpha_i(a_i)$  is the probability that  $a_i$  will be chosen in the mixed strategy  $\alpha_i$
- A profile  $\alpha = (\alpha_i)$  of mixed strategies induces a probability distribution on  $A$ :  $p(a) = \prod_i \alpha_i(a_i)$
- The expected utility is  $U_i(\alpha) = \sum_{a \in A} p(a) u_i(a)$

# Example of a Mixed Strategy

- Let
  - $\alpha_1(H) = 2/3, \alpha_1(T) = 1/3$
  - $\alpha_2(H) = 1/3, \alpha_2(T) = 2/3$
- Then
  - $p(H,H) = 2/9$
  - $p(H,T) =$
  - $p(T,H) =$
  - $p(T,T) =$
  - $U_1(\alpha_1, \alpha_2) =$

|      | Head | Tail |
|------|------|------|
| Head | 1,-1 | -1,1 |
| Tail | -1,1 | 1,-1 |

# Mixed Extensions

- The **mixed extension** of the strategic game  $(N, (A_i), (u_i))$  is the strategic game  $(N, \Delta(A_i), (U_i))$ .
- The **mixed strategy Nash equilibrium of a strategic game** is a Nash equilibrium of its mixed extension.
- Note that the **Nash equilibria in pure strategies** (as studied in the last part) are just a special case of mixed strategy equilibria.

# Nash's Theorem

**Theorem.** Every finite strategic game has a mixed strategy Nash equilibrium.

- Note that it is essential that the game is **finite**
- So, there **exists** always a solution
- What is the **computational complexity**?
- **Identifying** a NE with a value larger than a particular value is **NP-hard**

# The Support

- We call all pure actions  $a_i$  that are chosen with non-zero probability by  $\alpha_i$  the **support** of the mixed strategy  $\alpha_i$

**Lemma.** Given a finite strategic game,  $\alpha^*$  is a *mixed strategy equilibrium* if and only if for every player  $i$  every pure strategy in the support of  $\alpha_i^*$  is a **best response** to  $\alpha_{-i}^*$ .

# Using the Support Lemma

- The **Support Lemma** can be used to compute all types of Nash equilibria in 2-person 2x2 action games.
  - There are 4 potential Nash equilibria in **pure strategies**
    - ❖ *Easy to check*
  - There are another 4 potential Nash equilibrium types with a **1-support** (pure) against **2-support** mixed strategies
    - ❖ Exists only if the **corresponding pure strategy profiles** are already Nash equilibria (follows from **Support Lemma**)
  - There exists one other potential Nash equilibrium type with a **2-support** against a **2-support** mixed strategies
    - ❖ Here we can use the **Support Lemma** to compute an NE (if there exists one)

# A Mixed Nash Equilibrium for Matching Pennies

|      | Head | Tail |
|------|------|------|
| Head | 1,-1 | -1,1 |
| Tail | -1,1 | 1,-1 |

- There is clearly no NE in **pure strategies**
- Lets try whether there is a **NE**  $\alpha^*$  in **mixed strategies**
- Then the H action by player 1 should have the same utility as the T action when played against the mixed strategy  $\alpha_{-1}^*$

- $U_1((1,0), (\alpha_2(H), \alpha_2(T))) = U_1((0,1), (\alpha_2(H), \alpha_2(T)))$
- $U_1((1,0), (\alpha_2(H), \alpha_2(T))) = 1\alpha_2(H) + -1\alpha_2(T)$
- $U_1((0,1), (\alpha_2(H), \alpha_2(T))) = -1\alpha_2(H) + 1\alpha_2(T)$
- $\alpha_2(H) - \alpha_2(T) = -\alpha_2(H) + \alpha_2(T)$
- $2\alpha_2(H) = 2\alpha_2(T)$
- $\alpha_2(H) = \alpha_2(T)$
- Because of  $\alpha_2(H) + \alpha_2(T) = 1$ :
  - $\alpha_2(H) = \alpha_2(T) = 1/2$
  - Similarly for player 1!
- ❖  $U_1(\alpha^*) = 0$

# Mixed NE for BoS

|                 | Bach | Stra-<br>vinsky |
|-----------------|------|-----------------|
| Bach            | 2,1  | 0,0             |
| Stra-<br>vinsky | 0,0  | 1,2             |

- There are obviously 2 NEs in pure strategies
- Is there also a strictly mixed NE?
- If so, again B and S played by player 1 should lead to the same payoff.

- $U_1((1,0), (\alpha_2(B), \alpha_2(S))) = U_1((0,1), (\alpha_2(B), \alpha_2(S)))$
- $U_1((1,0), (\alpha_2(B), \alpha_2(S))) = 2\alpha_2(B) + 0\alpha_2(S)$
- $U_1((0,1), (\alpha_2(B), \alpha_2(S))) = 0\alpha_2(B) + 1\alpha_2(S)$
- $2\alpha_2(B) = 1\alpha_2(S)$
- Because of  $\alpha_2(B) + \alpha_2(S) = 1$ :
  - $\alpha_2(B) = 1/3$
  - $\alpha_2(S) = 2/3$
- Similarly for player 1!
- ❖  $U_1(\alpha^*) = 2/3$

# The 2/3 of Average Game

- You have  $n$  players that are allowed to choose a number between 1 and  $K$ .
- The players coming **closest to 2/3 of the average** over all numbers win. A fixed prize is **split equally** between all the winners
- What number would **you** play?
- What **mixed strategy** would you play?

# A Nash Equilibrium in Pure Strategies

- All playing 1 is a NE in pure strategies
  - A deviation does not make sense
- All playing the same number different from 1 is **not a NE**
  - Choosing the number just below gives you more
- Similar, when all play different numbers, some not winning anything could get closer to  $2/3$  of the average and win something.
- So: ***Why did you not choose 1?***
- Perhaps **you acted rationally** by assuming that the **others do not act rationally?**

# Are there Proper Mixed Strategy Nash Equilibria?

- Assume there exists a mixed NE  $\alpha$  different from the pure NE  $(1, 1, \dots, 1)$
- Then there exists a maximal  $k^* > 1$  which is played by some player with a probability  $> 0$ .
  - Assume player  $i$  does so, i.e.,  $k^*$  is in the support of  $\alpha_i$ .
- This implies  $U_i(k^*, \alpha_{-i}) > 0$ , since  $k^*$  should be as good as all the other strategies of the support.
- Let  $a$  be a realization of  $\alpha$  s.t.  $u_i(a) > 0$ . Then at least one other player must play  $k^*$ , because not all others could play below  $2/3$  of the average!
- In this situation player  $i$  could get more by playing  $k^*-1$ .
- This means, playing  $k^*-1$  is better than playing  $k^*$ , i.e.,  $k^*$  cannot be in the support, i.e.,  **$\alpha$  cannot be a NE**

# Summary

- **Strategic games** are one-shot games, where everybody plays its move simultaneously
- Each player gets a payoff based on its **payoff function** and the resulting **action profile**.
- **Iterated elimination of strictly dominated strategies** is a convincing solution concept.
- **Nash equilibrium** is another solution concept: Action profiles, where **no player has an incentive to deviate**
- It also might **not be unique** and there can be even infinitely many NEs or none at all!
- For every finite strategic game, there exists a Nash equilibrium in **mixed strategies**
- Actions in the support of mixed strategies in a NE are always best answers to the NE profile, and therefore have the same payoff □ **Support Lemma**
- Computing a NE in mixed strategies is NP-hard