Foundations of AI

18. Strategic Games

Strategic Reasoning and Acting

Wolfram Burgard and Bernhard Nebel
Strategic Game

• A strategic game $G$ consists of
  – a finite set $N$ (the set of players)
  – for each player $i \in N$ a non-empty set $A_i$ (the set of actions or strategies available to player $i$), whereby $A = \prod_i A_i$
  – for each player $i \in N$ a function $u_i: A \rightarrow R$ (the utility or payoff function)
  – $G = (N, (A_i), (u_i))$

• If $A$ is finite, then we say that the game is finite
Playing the Game

- Each player $i$ makes a decision which action to play: $a_i$
- All players make their moves simultaneously leading to the action profile $a^* = (a_1, a_2, \ldots, a_n)$
- Then each player gets the payoff $u_i(a^*)$
- Of course, each player tries to maximize its own payoff, but what is the right decision?
- **Note:** While we want to maximize our payoff, we are not interested in harming our opponent. It just does not matter to us what he will get!
  - If we want to model something like this, the payoff function must be changed
Notation

- For 2-player games, we use a matrix, where the strategies of player 1 are the rows and the strategies of player 2 the columns.
- The payoff for every action profile is specified as a pair $x,y$, whereby $x$ is the value for player 1 and $y$ is the value for player 2.
- Example: For (T,R), player 1 gets $x_{12}$, and player 2 gets $y_{12}$.
Example Game: Bach and Stravinsky

- Two people want to go together to a concert of music by either Bach or Stravinsky. Their main concern is to go out together, but one prefers Bach, the other Stravinsky. Will they meet?
- This game is also called the *Battle of the Sexes*

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Example Game: Hawk-Dove

- Two animals fighting over some prey.
- Each can behave like a dove or a hawk.
- The best outcome is if oneself behaves like a hawk and the opponent behaves like a dove.
- This game is also called chicken.

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Example Game: Prisoner’s Dilemma

- Two suspects in a crime are put into separate cells.
- If they both confess, each will be sentenced to 3 years in prison.
- If only one confesses, he will be freed.
- If neither confesses, they will both be convicted of a minor offense and will spend one year in prison.

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Solving a Game

• What is the right move?

• Different possible solution concepts
  – Elimination of strictly or weakly dominated strategies
  – Maximin strategies (for minimizing the loss in zero-sum games)
  – Nash equilibrium

• How difficult is it to compute a solution?

• Are there always solutions?

• Are the solutions unique?
Strictly Dominated Strategies

• Notation:
  – Let $a = (a_i)$ be a strategy profile
  – $a_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n)$
  – $(a_{-i}, a'_i) := (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots a_n)$

• Strictly dominated strategy:
  – An strategy $a_j^* \in A_j$ is \textit{strictly dominated} if there exists a strategy $a'_j$ such that for all strategy profiles $a \in A$:
    \[ u_j(a_{-j}, a'_j) > u_j(a_{-j}, a_j^*) \]

• Of course, it is \textbf{not rational} to play strictly dominated strategies
Iterated Elimination of Strictly Dominated Strategies

- Since strictly dominated strategies will never be played, one can eliminate them from the game.
- This can be done iteratively.
- If this converges to a single strategy profile, the result is unique.
- This can be regarded as the result of the game, because it is the only rational outcome.
## Iterated Elimination: Example

- **Eliminate:**
  - dominated by
  - dominated by
  - dominated by
  - dominated by

- **Result:**

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<tr>
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<th>b1</th>
<th>b2</th>
<th>b3</th>
<th>b4</th>
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<tbody>
<tr>
<td>a1</td>
<td>1,7</td>
<td>2,5</td>
<td>7,2</td>
<td>0,1</td>
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<tr>
<td>a2</td>
<td>5,2</td>
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<tr>
<td>a3</td>
<td>7,0</td>
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<td>0,4</td>
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<td>a4</td>
<td>0,0</td>
<td>0,-2</td>
<td>0,0</td>
<td>9,-1</td>
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Iterated Elimination: Prisoner’s Dilemma

• Player 1 reasons that “not confessing” is strictly dominated and eliminates this option
• Player 2 reasons that player 1 will not consider “not confessing”. So he will eliminate this option for himself as well
• So, they both confess

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Weakly Dominated Strategies

• Instead of strict domination, we can also go for weak domination:
  – An strategy $a_j^* \in A_j$ is \textit{weakly dominated} if there exists a strategy $a_j'$ such that for all strategy profiles $a \in A$:
    \[
    u_j(a_{-j}, a_j) \geq u_j(a_{-j}, a_j^*)
    \]
    and for at least one profile $a \in A$:
    \[
    u_j(a_{-j}, a_j') > u_j(a_{-j}, a_j^*).
    \]
Results of Iterative Elimination of Weakly Dominated Strategies

• The result is not necessarily unique

• Example:
  – Eliminate
    • T ($\leq M$)
    • L ($\leq R$)

    Result: (1,1)
  – Eliminate:
    • B ($\leq M$)
    • R ($\leq L$)

    Result: (2,1)

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<td>T</td>
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<td>M</td>
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Analysis of the Guessing 2/3 of the Average Game

• All strategies above 67 are weakly dominated, since they will *never ever* lead to winning the prize, so they can be eliminated!
• This means, that all strategies above \( \frac{2}{3} \times 67 \) can be eliminated
• … and so on
• … until all strategies above 1 have been eliminated!
• So: The rationale strategy would be to play 1!
Existence of Dominated Strategies

• Dominating strategies are a convincing solution concept

• Unfortunately, often dominated strategies do not exist

• What do we do in this case?

  ➢ Nash equilibrium

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Nash Equilibrium

• A *Nash equilibrium* is an action profile $a^* \in A$ with the property that for all players $i \in N$:
  $$u_i(a^*) = u_i(a^*_{-i}, a^*_i) \geq u_i(a^*_{-i}, a_i) \forall a_i \in A_i$$

• In words, it is an action profile such that there is no incentive for any agent to deviate from it

• While it is less convincing than an action profile resulting from iterative elimination of dominated strategies, it is still a reasonable solution concept

• If there exists a unique solution from iterated elimination of strictly dominated strategies, then it is also a *Nash equilibrium*
Example Nash-Equilibrium: Prisoner’s Dilemma

- Don’t – Don’t
  - not a NE
- Don’t – Confess (and vice versa)
  - not a NE
- Confess – Confess
  - NE

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Example Nash-Equilibrium: Hawk-Dove

- Dove-Dove:  
  - not a NE
- Hawk-Hawk  
  - not a NE
- Dove-Hawk  
  - is a NE
- Hawk-Dove  
  - is, of course, another NE
- So, NEs are not necessarily unique

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Auctions

- An **object** is to be **assigned** to a player in the set \( \{1, \ldots, n\} \) in exchange for a payment.
- Players’ **valuation** of the object is \( v_i \), and \( v_1 > v_2 > \ldots > v_n \).
- The mechanism to assign the object is a **sealed-bid auction**: the players simultaneously submit bids (non-negative real numbers).
- The object is given to the player with the lowest index among those who submit the highest bid in exchange for the payment.
- The payment for a **first price** auction is the highest bid.
- What are the Nash equilibria in this case?
Formalization

• Game \( G = (\{1,\ldots,n\}, (A_i), (u_i)) \)
• \( A_i \): bids \( b_i \in \mathbb{R}^+ \)
• \( u_i(b_{-i}, b_i) = v_i - b_i \) if \( i \) has won the auction, 0 otherwise
• Nobody would bid more than his valuation, because this could lead to negative utility, and we could easily achieve 0 by bidding 0.
Nash Equilibria for First-Price Sealed-Bid Auctions

• The Nash equilibria of this game are all profiles $b$ with:
  
  \(- b_i \leq b_1 \text{ for all } i \in \{2, \ldots, n\}\)

  \(- \text{ No } i \text{ would bid more than } v_2 \text{ because it could lead to negative utility}\)

  \(- \text{ If a } b_i \text{ (with } < v_2\text{) is higher than } b_1 \text{ player 1 could increase its utility by bidding } v_2 + \varepsilon\)

  \(- \text{ So 1 wins in all NEs}\)

\(- v_1 \geq b_1 \geq v_2\)

\(- \text{ Otherwise, player 1 either looses the bid (and could increase its utility by bidding more) or would have itself negative utility}\)

\(- b_j = b_1 \text{ for at least one } j \in \{2, \ldots, n\}\)

\(- \text{ Otherwise player 1 could have gotten the object for a lower bid}\)
Another Game: Matching Pennies

- Each of two people chooses either Head or Tail. If the choices differ, player 1 pays player 2 a euro; if they are the same, player 2 pays player 1 a euro.
- This is also a zero-sum or strictly competitive game.
- No NE at all! What shall we do here?

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Randomizing Actions …

- Since there does not seem to exist a rational decision, it might be best to randomize strategies.
- Play Head with probability $p$ and Tail with probability $1-p$.
- Switch to expected utilities.

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Some Notation

- Let $G = (N, (A_i), (u_i))$ be a strategic game.
- Then $\Delta(A_i)$ shall be the set of probability distributions over $A_i$—the set of mixed strategies $\alpha_i \in \Delta(A_i)$.
- $\alpha_i(a_i)$ is the probability that $a_i$ will be chosen in the mixed strategy $\alpha_i$.
- A profile $\alpha = (\alpha_i)$ of mixed strategies induces a probability distribution on $A$: $p(a) = \prod_i \alpha_i(a_i)$.
- The expected utility is $U_i(\alpha) = \sum_{a \in A} p(a) u_i(a)$. 
Example of a Mixed Strategy

Let
- \( \alpha_1(H) = \frac{2}{3} \), \( \alpha_1(T) = \frac{1}{3} \)
- \( \alpha_2(H) = \frac{1}{3} \), \( \alpha_2(T) = \frac{2}{3} \)

Then
- \( p(H,H) = \frac{2}{9} \)
- \( p(H,T) = \)
- \( p(T,H) = \)
- \( p(T,T) = \)
- \( U_1(\alpha_1, \alpha_2) = \)

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Mixed Extensions

• The mixed extension of the strategic game $(N, (A_i), (u_i))$ is the strategic game $(N, \Delta(A_i), (U_i))$.

• The mixed strategy Nash equilibrium of a strategic game is a Nash equilibrium of its mixed extension.

• Note that the Nash equilibria in pure strategies (as studied in the last part) are just a special case of mixed strategy equilibria.
Nash’s Theorem

**Theorem.** Every finite strategic game has a mixed strategy Nash equilibrium.

- Note that it is essential that the game is finite
- So, there exists always a solution
- What is the computational complexity?
- Identifying a NE with a value larger than a particular value is NP-hard
The Support

• We call all pure actions $a_i$ that are chosen with non-zero probability by $\alpha_i$ the support of the mixed strategy $\alpha_i$.

**Lemma.** Given a finite strategic game, $\alpha^*$ is a mixed strategy equilibrium if and only if for every player $i$ every pure strategy in the support of $\alpha_i^*$ is a best response to $\alpha_{-i}^*$. 


Using the Support Lemma

- The **Support Lemma** can be used to compute all types of Nash equilibria in 2-person 2x2 action games.
- There are 4 potential Nash equilibria in **pure strategies**
  - *Easy to check*
- There are another 4 potential Nash equilibrium types with a **1-support** (pure) against **2-support** mixed strategies
  - Exists only if the corresponding pure strategy profiles are already Nash equilibria (follows from **Support Lemma**)
- There exists one other potential Nash equilibrium type with a **2-support** against a **2-support** mixed strategies
  - Here we can use the **Support Lemma** to compute an NE (if there exists one)
A Mixed Nash Equilibrium for Matching Pennies

- There is clearly no NE in pure strategies.
- Let's try whether there is a NE $\alpha^*$ in mixed strategies.
- Then the H action by player 1 should have the same utility as the T action when played against the mixed strategy $\alpha_1^*$.

### Payoff Matrix

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- $U_1((1,0), (\alpha_2(H), \alpha_2(T))) = U_1((0,1), (\alpha_2(H), \alpha_2(T)))$
- $U_1((1,0), (\alpha_2(H), \alpha_2(T))) = 1\alpha_2(H) + -1\alpha_2(T)$
- $U_1((0,1), (\alpha_2(H), \alpha_2(T))) = -1\alpha_2(H) + 1\alpha_2(T)$
- $\alpha_2(H) - \alpha_2(T) = -\alpha_2(H) + \alpha_2(T)$
- $2\alpha_2(H) = 2\alpha_2(T)$
- $\alpha_2(H) = \alpha_2(T)$
- Because of $\alpha_2(H) + \alpha_2(T) = 1$:
  - $\alpha_2(H) = \alpha_2(T) = 1/2$
  - Similarly for player 1!

- $U_1(\alpha^*) = 0$
Mixed NE for BoS

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- There are obviously 2 NEs in pure strategies
- Is there also a strictly mixed NE?
- If so, again B and S played by player 1 should lead to the same payoff.

\[ U_1((1,0), (\alpha_2(B), \alpha_2(S))) = U_1((0,1), (\alpha_2(B), \alpha_2(S))) \]

\[ U_1((1,0), (\alpha_2(B), \alpha_2(S))) = 2\alpha_2(B) + 0\alpha_2(S) \]
\[ U_1((0,1), (\alpha_2(B), \alpha_2(S))) = 0\alpha_2(B) + 1\alpha_2(S) \]
\[ 2\alpha_2(B) = 1\alpha_2(S) \]

Because of \( \alpha_2(B) + \alpha_2(S) = 1 \):
- \( \alpha_2(B) = 1/3 \)
- \( \alpha_2(S) = 2/3 \)

- Similarly for player 1!

- \( U_1(\alpha^*) = 2/3 \)
The 2/3 of Average Game

- You have $n$ players that are allowed to choose a number between 1 and $K$.
- The players coming closest to 2/3 of the average over all numbers win. A fixed prize is split equally between all the winners.
- What number would you play?
- What mixed strategy would you play?
A Nash Equilibrium in Pure Strategies

- All playing 1 is a NE in pure strategies
  - A deviation does not make sense
- All playing the same number different from 1 is not a NE
  - Choosing the number just below gives you more
- Similar, when all play different numbers, some not winning anything could get closer to 2/3 of the average and win something.

So: *Why did you not choose 1?*

- Perhaps you acted rationally by assuming that the others do not act rationally?
Are there Proper Mixed Strategy Nash Equilibria?

• Assume there exists a mixed NE \( \alpha \) different from the pure NE \((1,1,...,1)\)
• Then there exists a maximal \( k^* > 1 \) which is played by some player with a probability > 0.
  – Assume player \( i \) does so, i.e., \( k^* \) is in the support of \( \alpha_i \).
• This implies \( U_i(k^*, \alpha_{-i}) > 0 \), since \( k^* \) should be as good as all the other strategies of the support.
• Let \( a \) be a realization of \( \alpha \) s.t. \( u_i(a) > 0 \). Then at least one other player must play \( k^* \), because not all others could play below \( 2/3 \) of the average!
• In this situation player \( i \) could get more by playing \( k^*-1 \).
• This means, playing \( k^*-1 \) is better than playing \( k^* \), i.e., \( k^* \) cannot be in the support, i.e., \( \alpha \) cannot be a NE
Summary

• **Strategic games** are one-shot games, where everybody plays its move simultaneously.
• Each player gets a payoff based on its payoff function and the resulting action profile.
• **Iterated elimination of strictly dominated strategies** is a convincing solution concept.
• **Nash equilibrium** is another solution concept: Action profiles, where no player has an incentive to deviate.
• It also might **not be unique** and there can be even infinitely many NEs or none at all!

➢ For every finite strategic game, there exists a Nash equilibrium in mixed strategies.
• Actions in the support of mixed strategies in a NE are always best answers to the NE profile, and therefore have the same payoff  
  \[ \text{Support Lemma} \]
• Computing a NE in mixed strategies is NP-hard.