

Introduction to Mobile Robotics

Error Propagation

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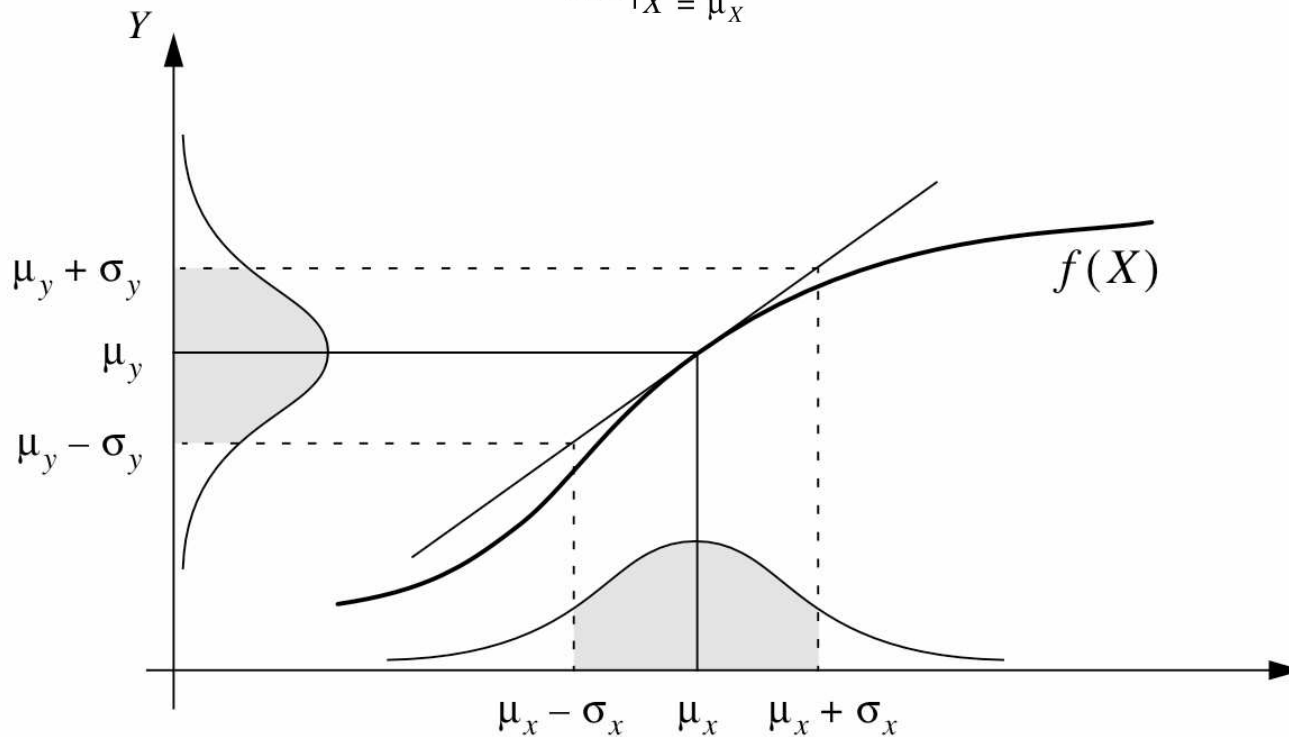
Error Propagation: Motivation

- Probabilistic robotics is
 - **Representation**
 - **Propagation**
 - **Reduction**
of uncertainty
- **First-order error propagation** is fundamental for:
Kalman filter (KF), landmark extraction,
KF-based localization and SLAM

First-Order Error Propagation

Approximating $f(X)$ by a **first-order** Taylor series expansion about the point $X = \mu_X$

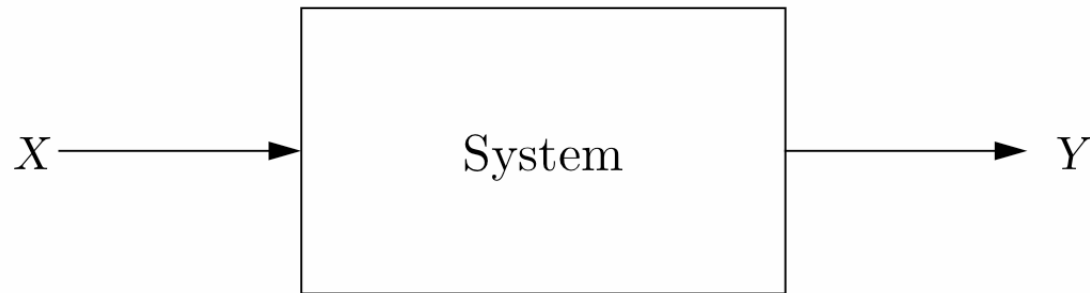
$$Y \approx f(\mu_X) + \left. \frac{\partial f}{\partial X} \right|_{X = \mu_X} (X - \mu_X)$$



First-Order Error Propagation

X, Y assumed to be Gaussian

$$Y = f(X)$$



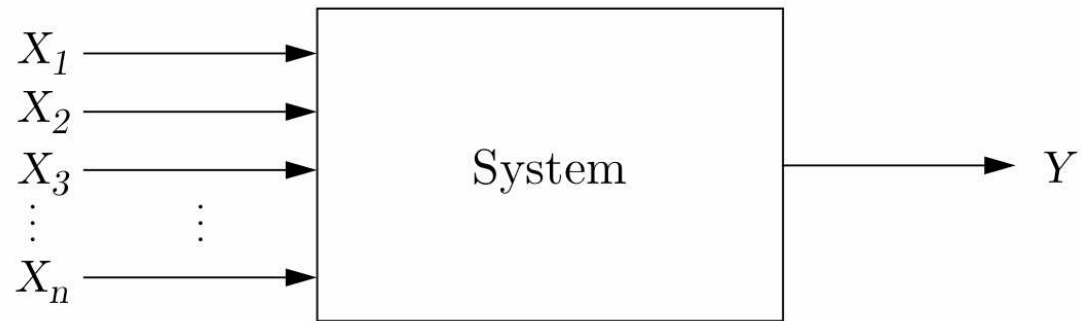
Taylor series expansion

$$Y \approx f(\mu_X) + \left. \frac{\partial f}{\partial X} \right|_{X = \mu_X} (X - \mu_X)$$

Wanted: μ_Y, σ_Y^2 (Solution on blackboard)

First-Order Error Propagation

$$Y = f(X_1, X_2, \dots, X_n)$$



Taylor series expansion

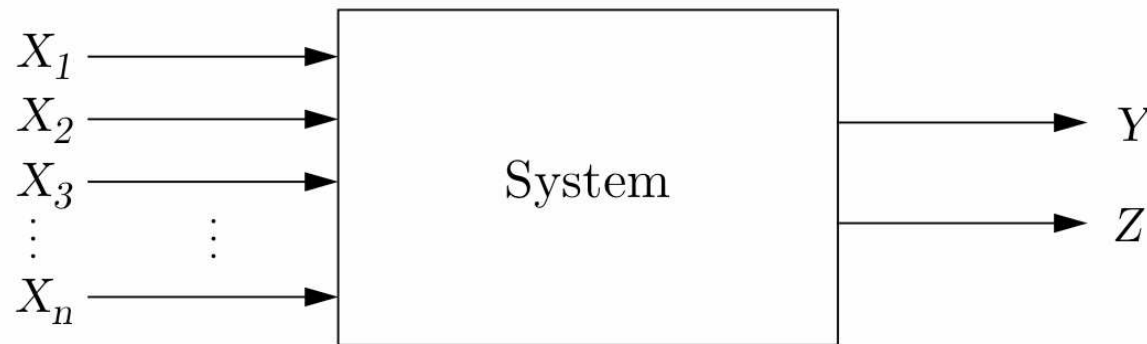
$$Y \approx f(\mu_1, \mu_2, \dots, \mu_n) + \sum_{i=1}^n \left[\frac{\partial f}{\partial X_i} (\mu_1, \mu_2, \dots, \mu_n) \right] [X_i - \mu_i]$$

Wanted: μ_Y , σ_Y^2 (Solution on blackboard)

First-Order Error Propagation

$$Y = f(X_1, X_2, \dots, X_n)$$

$$Z = g(X_1, X_2, \dots, X_n)$$

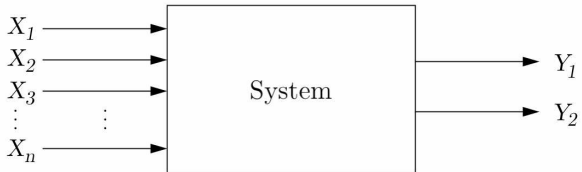


Wanted: σ_{YZ}

(Exercise)

First-Order Error Propagation

Putting things together...

$$C_X = \begin{bmatrix} \sigma_{X_1}^2 & \sigma_{X_1X_2} & \cdots & \sigma_{X_1X_n} \\ \sigma_{X_2X_1} & \sigma_{X_2}^2 & \cdots & \sigma_{X_2X_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{X_nX_1} & \sigma_{X_nX_2} & \cdots & \sigma_{X_n}^2 \end{bmatrix}$$


$$C_Y = \begin{bmatrix} \sigma_{Y_1}^2 & \sigma_{Y_1Y_2} \\ \sigma_{Y_2Y_1} & \sigma_{Y_2}^2 \end{bmatrix}$$

with

$$\sigma_Y^2 = \sum_i \left(\frac{\partial f}{\partial X_i} \right)^2 \sigma_i^2 + \sum_{i \neq j} \left(\frac{\partial f}{\partial X_i} \right) \left(\frac{\partial f}{\partial X_j} \right) \sigma_{ij}$$

$$\sigma_{YZ} = \sum \left(\frac{\partial f}{\partial X_i} \right) \left(\frac{\partial g}{\partial X_i} \right) \sigma_i^2 + \sum_{i \neq j} \left(\frac{\partial f}{\partial X_i} \right) \left(\frac{\partial g}{\partial X_j} \right) \sigma_{ij}$$

→ “Is there a **compact form?...**”

Jacobian Matrix

- It's a **non-square matrix** $n \times m$ in general
- Suppose you have a vector-valued function $f(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix}$
- Let the **gradient operator** be the vector of (first-order) partial derivatives

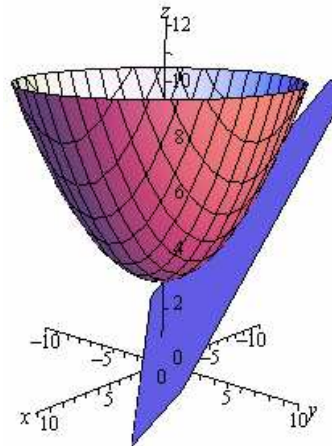
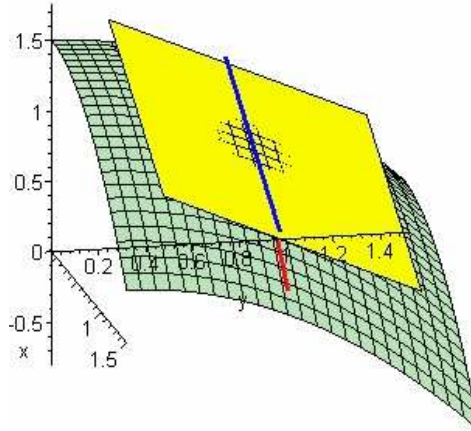
$$\nabla_{\mathbf{x}} = \left[\frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \quad \cdots \quad \frac{\partial}{\partial x_n} \right]^T$$

- Then, the **Jacobian matrix** is defined as

$$\mathbf{F}_{\mathbf{x}} = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix} \cdot \left[\frac{\partial}{\partial x_1} \quad \cdots \quad \frac{\partial}{\partial x_n} \right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \end{bmatrix}$$

Jacobian Matrix

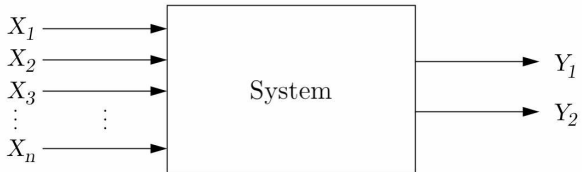
- It's the orientation of the **tangent plane** to the vector-valued function at a given point



- **Generalizes the gradient** of a scalar valued function
- Heavily used for **first-order error propagation...**

First-Order Error Propagation

Putting things together...

$$C_X = \begin{bmatrix} \sigma_{X_1}^2 & \sigma_{X_1X_2} & \cdots & \sigma_{X_1X_n} \\ \sigma_{X_2X_1} & \sigma_{X_2}^2 & \cdots & \sigma_{X_2X_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{X_nX_1} & \sigma_{X_nX_2} & \cdots & \sigma_{X_n}^2 \end{bmatrix}$$


$$C_Y = \begin{bmatrix} \sigma_{Y_1}^2 & \sigma_{Y_1Y_2} \\ \sigma_{Y_2Y_1} & \sigma_{Y_2}^2 \end{bmatrix}$$

with

$$\sigma_Y^2 = \sum_i \left(\frac{\partial f}{\partial X_i} \right)^2 \sigma_i^2 + \sum_{i \neq j} \left(\frac{\partial f}{\partial X_i} \right) \left(\frac{\partial f}{\partial X_j} \right) \sigma_{ij}$$

$$\sigma_{YZ} = \sum \left(\frac{\partial f}{\partial X_i} \right) \left(\frac{\partial g}{\partial X_i} \right) \sigma_i^2 + \sum_{i \neq j} \left(\frac{\partial f}{\partial X_i} \right) \left(\frac{\partial g}{\partial X_j} \right) \sigma_{ij}$$

→ “Is there a **compact form?...**”

First-Order Error Propagation

...**Yes!** Given

- Input covariance matrix C_X
- Jacobian matrix F_X

the **Error Propagation Law**

$$C_Y = F_X C_X F_X^T$$

computes the output covariance matrix C_Y

First-Order Error Propagation

Alternative Derivation:

$$\begin{aligned}\mu_x &= E(x) \\ &= E(Au + b) \\ &= AE(u) + b \\ &= A\mu_u + b\end{aligned}$$

$$\begin{aligned}\Sigma_x &= E((x - E(x))(x - E(x))^T) \\ &= E((Au + b - AE(u) - b)(Au + b - AE(u) - b)^T) \\ &= E((A(u - E(u)))(A(u - E(u)))^T) \\ &= E((A(u - E(u))((u - E(u))^T A^T)) \\ &= AE((u - E(u))(u - E(u))^T)A^T \\ &= A\Sigma_u A^T\end{aligned}$$

Example: Line Extraction

Wanted: Parameter Covariance Matrix

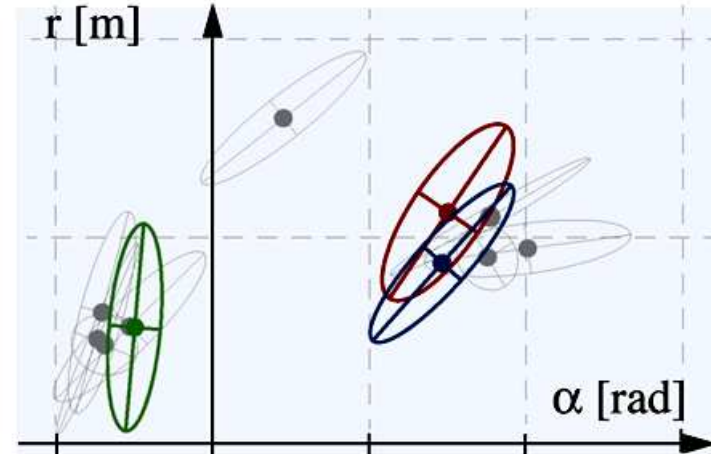
$$C_{AR} = \begin{bmatrix} \sigma_A^2 & \sigma_{AR} \\ \sigma_{AR} & \sigma_R^2 \end{bmatrix}$$

$$C_X = \begin{bmatrix} \sigma_{\rho_1}^2 & 0 & \dots & 0 \\ 0 & \sigma_{\rho_2}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{\rho_n}^2 \end{bmatrix}$$

Simplified sensor model:
all $\sigma_{\theta_i}^2 = 0$, independence

$$C_{AR} = F_X C_X F_X^T$$

Result: Gaussians in the model space



Other Error Prop. Techniques

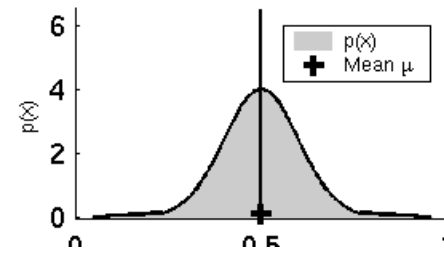
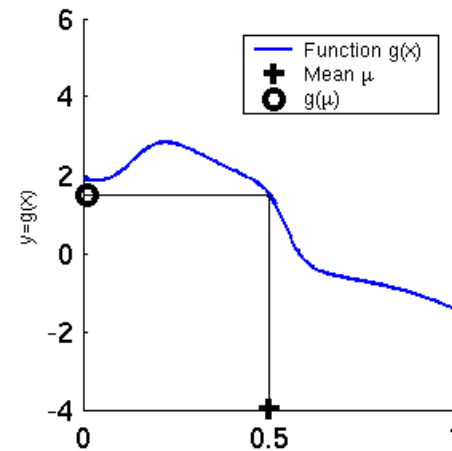
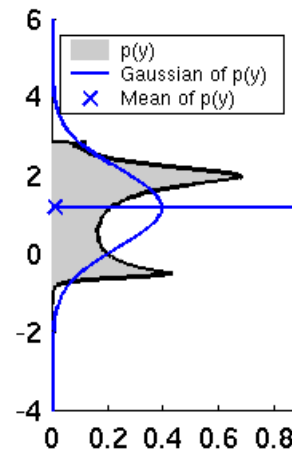
- **Second-Order Error Propagation**

Rarely used (complex expressions)

- **Monte-Carlo**

Non-parametric representation of uncertainties

1. Propagation of samples
2. Histogramming
3. Normalization



Derivations (2/4)

Definitions

$$\mu = E(X)$$

$$\text{Var}(X) = E((X - \mu)^2)$$

$$\text{Cov}(X, Y) = E((X - \mu)(Y - \nu))$$

Rules

$$E(X + c) = E(X) + c$$

$$E(X + Y) = E(X) + E(Y)$$

$$E(aX) = aE(X)$$

Result SISO

$$\mu_Y = f(\mu_X),$$

$$\sigma_Y = \left. \frac{\partial f}{\partial X} \right|_{X = \mu_X} \sigma_X.$$

Derivations (3/4)

Result MISO

$$\begin{aligned}\mu_Y &= E[Y] = E[a_0 + \sum_i a_i(X_i - \mu_i)] \\ &= E[a_0] + \sum_i E[a_i X_i] - E[a_i \mu_i] \\ &= a_0 + \sum_i a_i E[X_i] - a_i E[\mu_i] \\ &= a_0 + \sum_i a_i \mu_i - a_i \mu_i \\ &= a_0\end{aligned}$$

$$\mu_Y = f(\mu_1, \mu_2, \dots, \mu_n)$$

$$\begin{aligned}\sigma_Y^2 &= E[(Y - \mu_Y)^2] = E[(\sum_i a_i(X_i - \mu_i))^2] \\ &= E[\sum_i a_i(X_i - \mu_i) \sum_j a_j(X_j - \mu_j)] \\ &= E[\sum_i a_i^2(X_i - \mu_i)^2 + \sum_{i \neq j} \sum a_i a_j (X_i - \mu_i)(X_j - \mu_j)] \\ &= \sum_i a_i^2 E[(X_i - \mu_i)^2] + \sum_{i \neq j} \sum a_i a_j E[(X_i - \mu_i)(X_j - \mu_j)] \\ &= \sum_i a_i^2 \sigma_i^2 + \sum_{i \neq j} \sum a_i a_j \sigma_{ij} \\ \sigma_Y^2 &= \sum_i \left(\frac{\partial f}{\partial X_i}\right)^2 \sigma_i^2 + \sum_{i \neq j} \sum \left(\frac{\partial f}{\partial X_i}\right) \left(\frac{\partial f}{\partial X_j}\right) \sigma_{ij}\end{aligned}$$

Derivations (4/4)

Result MIMO

$$\begin{aligned}\sigma_{YZ} &= E[(Y - \mu_Y)(Z - \mu_Z)] \\ &= E[Y \cdot Z] - E[Y]E[Z] \\ &= E\left[\left(\mu_Y + \sum \frac{\partial f}{\partial X_i} [X_i - \mu_i]\right) \cdot \left(\mu_Z + \sum \frac{\partial g}{\partial X_i} [X_i - \mu_i]\right)\right] - \mu_Y \mu_Z \\ &= E\left[\mu_Y \mu_Z + \mu_Z \sum \frac{\partial f}{\partial X_i} [X_i - \mu_i] + \mu_Y \sum \frac{\partial g}{\partial X_i} [X_i - \mu_i] + \sum \frac{\partial f}{\partial X_i} [X_i - \mu_i] \sum \frac{\partial g}{\partial X_i} [X_i - \mu_i]\right] - \mu_Y \mu_Z \\ &= E[\mu_Y \mu_Z] + \mu_Z E\left[\sum \frac{\partial f}{\partial X_i} X_i - \sum \frac{\partial f}{\partial X_i} \mu_i\right] + \mu_Y E\left[\sum \frac{\partial g}{\partial X_i} X_i - \sum \frac{\partial g}{\partial X_i} \mu_i\right] \\ &\quad + E\left[\sum \sum \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} [X_i - \mu_i][X_j - \mu_j]\right] - \mu_Y \mu_Z \\ &= \mu_Y \mu_Z + \mu_Z \sum \frac{\partial f}{\partial X_i} E[X_i] - \mu_Z \sum \frac{\partial f}{\partial X_i} E[\mu_i] + \mu_Y \sum \frac{\partial g}{\partial X_i} E[X_i] - \mu_Y \sum \frac{\partial g}{\partial X_i} E[\mu_i] \\ &\quad + E\left[\sum \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_i} [X_i - \mu_i]^2 + \sum \sum_{i \neq j} \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} [X_i - \mu_i][X_j - \mu_j]\right] - \mu_Y \mu_Z \\ &= \sum \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_i} E[(X_i - \mu_i)^2] + \sum \sum_{i \neq j} \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} E[(X_i - \mu_i)(X_j - \mu_j)] \\ \sigma_{YZ} &= \sum \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_i} \sigma_i^2 + \sum \sum_{i \neq j} \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} \sigma_{ij}^2\end{aligned}$$