Introduction to Mobile Robotics

Error Propagation

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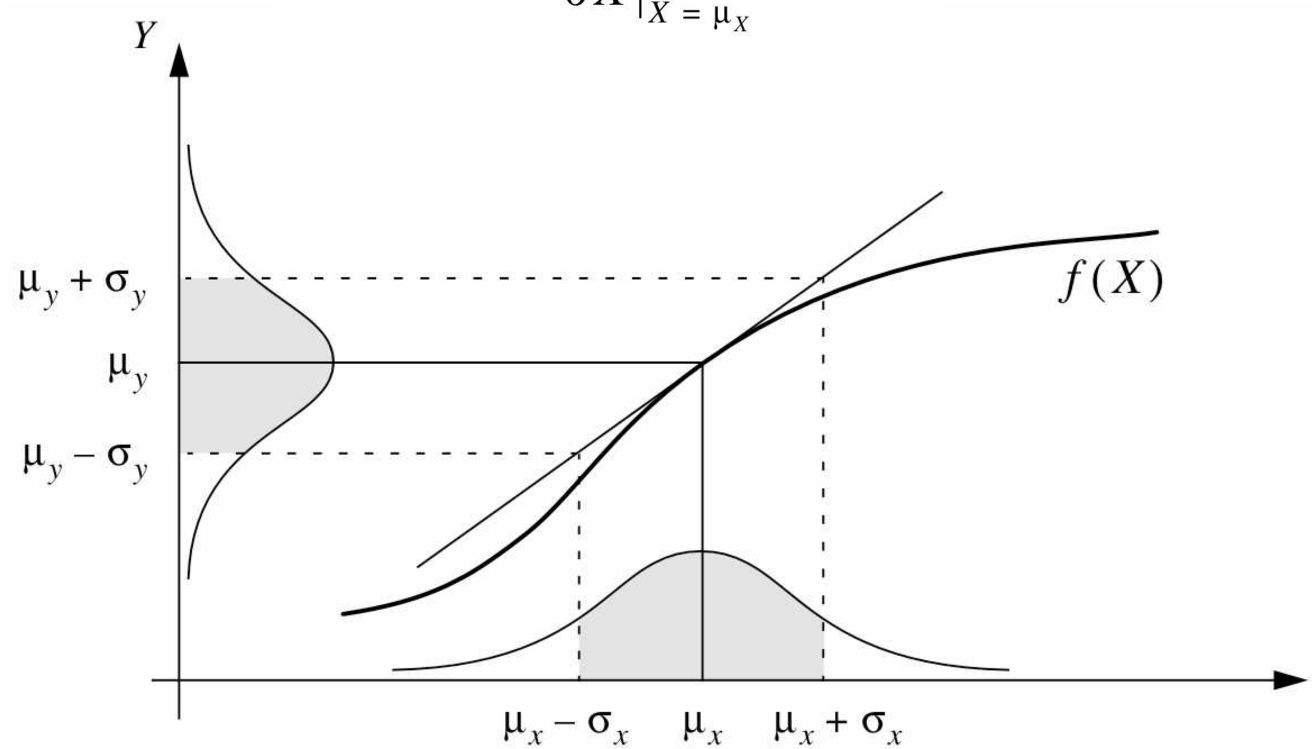
First-Order Error Propagation

Approximating $f(X)$ by a **first-order** Taylor series expansion about the point $X = \mu_X$

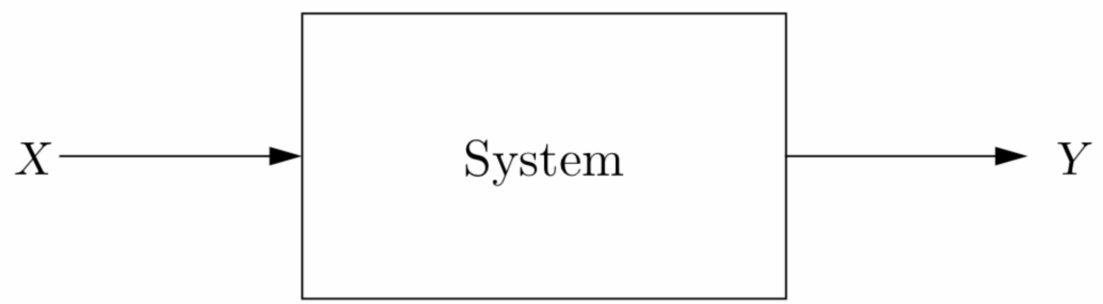
$$Y \approx f(\mu_X) + \left.\frac{\partial f}{\partial X}\right|_{X = \mu_X} (X - \mu_X)$$



First-Order Error Propagation

X,Y assumed to be Gaussian

$Y = f(X)$



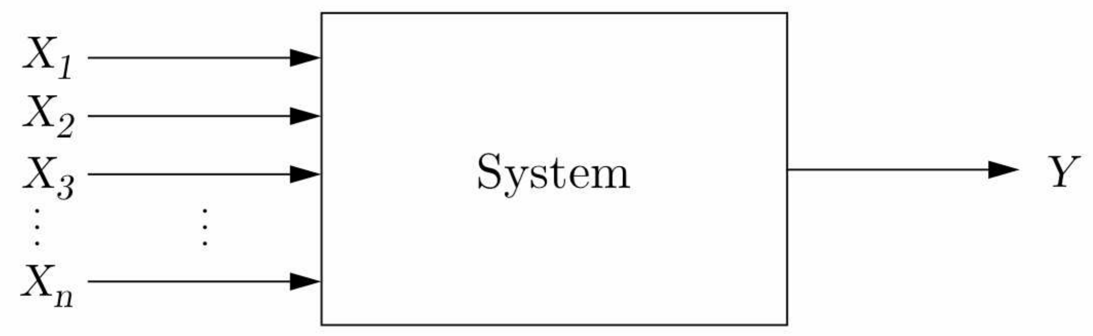
Taylor series expansion

$$Y \approx f(\mu_X) + \left.\frac{\partial f}{\partial X}\right|_{X = \mu_X} (X - \mu_X)$$

Wanted: μ_Y , σ_Y^2 (Solution on blackboard)

First-Order Error Propagation

$Y = f(X_1, X_2, \ldots, X_n)$



Taylor series expansion

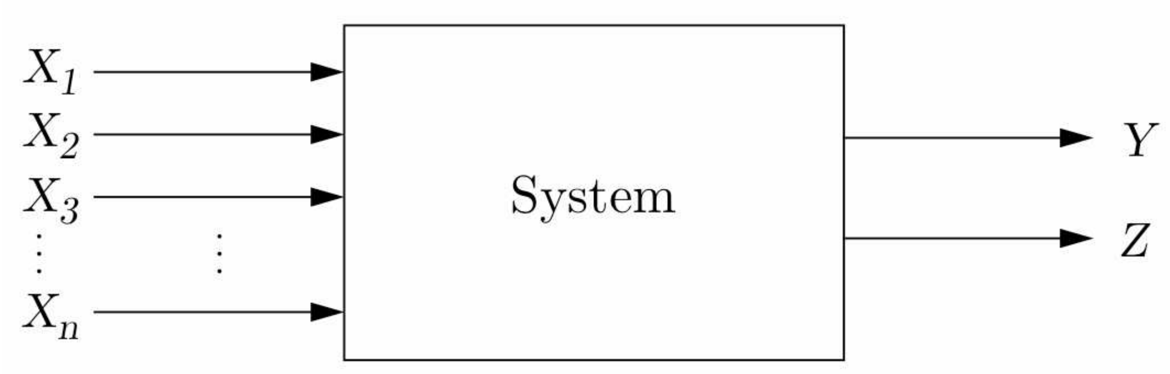
$$Y \approx f(\mu_1, \mu_2, \ldots, \mu_n) + \sum_{i=1}^{n} \left[\frac{\partial f}{\partial X_i}(\mu_1, \mu_2, \ldots, \mu_n) \right] [X_i - \mu_i]$$

Wanted: μ_Y, σ_Y^2 (Solution on blackboard)

First-Order Error Propagation

$Y = f(X_1, X_2, ..., X_n)$

$Z = g(X_1, X_2, ..., X_n)$



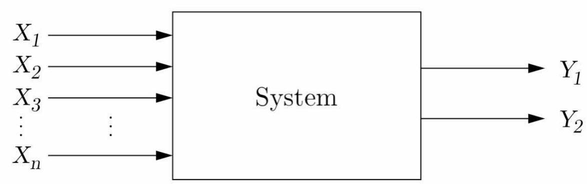
Wanted: σ_{YZ}

(Solution on blackboard)

First-Order Error Propagation

Putting things together...

$$C_X = \begin{bmatrix} \sigma_{X_1}^2 & \sigma_{X_1X_2} & \cdots & \sigma_{X_1X_n} \\ \sigma_{X_2X_1} & \sigma_{X_2}^2 & \cdots & \sigma_{X_2X_n} \\ \vdots & \vdots & & \vdots \\ \sigma_{X_nX_1} & \sigma_{X_nX_2} & \cdots & \sigma_{X_n}^2 \end{bmatrix}$$



$$C_Y = \begin{bmatrix} \sigma_{Y_1}^2 & \sigma_{Y_1Y_2} \\ \sigma_{Y_2Y_1} & \sigma_{Y_2}^2 \end{bmatrix}$$

with

$$\sigma_Y^2 = \sum_i \left(\frac{\partial f}{\partial X_i}\right)^2 \sigma_i^2 + \sum_{i \neq j}\sum \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial f}{\partial X_j}\right)\sigma_{ij}$$

$$\sigma_{YZ} = \sum \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial g}{\partial X_i}\right)\sigma_i^2 + \sum_{i \neq j}\sum \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial g}{\partial X_j}\right)\sigma_{ij}$$

→ “Is there a **compact form?...**”

Jacobian Matrix

- It's a **non-square matrix** $n \times m$ in general
- Suppose you have a vector-valued function $f(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix}$
- Let the **gradient operator** be the vector of (first-order) partial derivatives

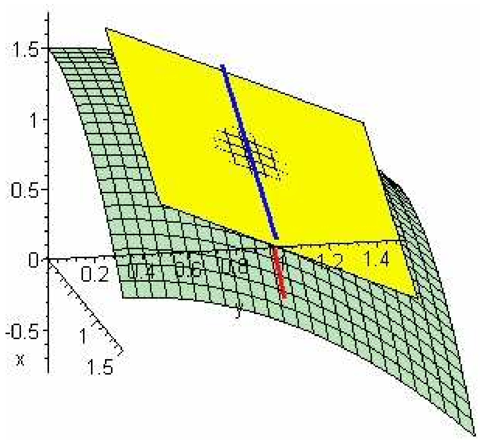
$$\nabla_{\mathbf{x}} = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n} \end{bmatrix}^T$$

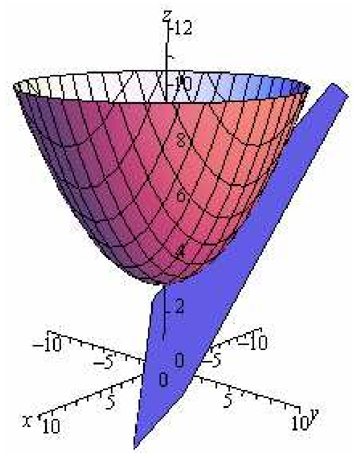
- Then, the **Jacobian matrix** is defined as

$$\mathbf{F}_{\mathbf{x}} = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \end{bmatrix}$$

Jacobian Matrix

- It's the orientation of the **tangent plane** to the vector-valued function at a given point



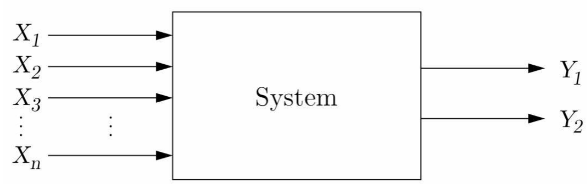


- **Generalizes the gradient** of a scalar valued function
- Heavily used for **first-order error propagation…**

First-Order Error Propagation

Putting things together...

$$C_X = \begin{bmatrix} \sigma_{X_1}^2 & \sigma_{X_1X_2} & \cdots & \sigma_{X_1X_n} \\ \sigma_{X_2X_1} & \sigma_{X_2}^2 & \cdots & \sigma_{X_2X_n} \\ \vdots & \vdots & & \vdots \\ \sigma_{X_nX_1} & \sigma_{X_nX_2} & \cdots & \sigma_{X_n}^2 \end{bmatrix}$$



$$C_Y = \begin{bmatrix} \sigma_{Y_1}^2 & \sigma_{Y_1Y_2} \\ \sigma_{Y_2Y_1} & \sigma_{Y_2}^2 \end{bmatrix}$$

with

$$\sigma_Y^2 = \sum_i \left(\frac{\partial f}{\partial X_i}\right)^2 \sigma_i^2 + \sum_{i \neq j}\sum \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial f}{\partial X_j}\right)\sigma_{ij}$$

$$\sigma_{YZ} = \sum \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial g}{\partial X_i}\right)\sigma_i^2 + \sum_{i \neq j}\sum \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial g}{\partial X_j}\right)\sigma_{ij}$$

→ "Is there a **compact form?...**"

First-Order Error Propagation

...Yes! Given

- Input covariance matrix C_X
- Jacobian matrix F_X

the **Error Propagation Law**

$$C_Y = F_X C_X F_X^T$$

computes the output covariance matrix C_Y

First-Order Error Propagation

Alternative Derivation:

$$
\begin{aligned}
\mu_x &= E(x) \\
&= E(Au + b) \\
&= AE(u) + b \\
&= A\mu_u + b
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_x &= E((x - E(x))(x - E(x))^T) \\
&= E((Au + b - AE(u) - b)(Au + b - AE(u) - b)^T) \\
&= E((A(u - E(u)))(A(u - E(u)))^T) \\
&= E((A(u - E(u)))((u - E(u))^T A^T)) \\
&= AE((u - E(u))(u - E(u))^T)A^T \\
&= A\Sigma_u A^T
\end{aligned}
$$

Example: Line Extraction

Wanted: Parameter Covariance Matrix

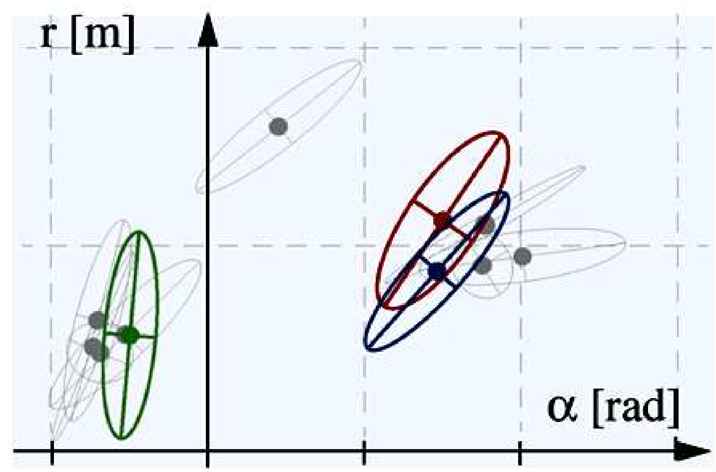
$$C_{AR} = \begin{bmatrix} \sigma_A^2 & \sigma_{AR} \\ \sigma_{AR} & \sigma_R^2 \end{bmatrix}$$

$$C_X = \begin{bmatrix} \sigma_{\rho_1}^2 & 0 & \dots & 0 \\ 0 & \sigma_{\rho_2}^2 & \dots & 0 \\ : & : & & : \\ 0 & 0 & \dots & \sigma_{\rho_n}^2 \end{bmatrix}$$

Simplified sensor model:
all $\sigma_{\theta_i}^2 = 0$, independence

$$C_{AR} = F_X C_X F_X^T$$

Result: Gaussians in the model space



Other Error Prop. Techniques

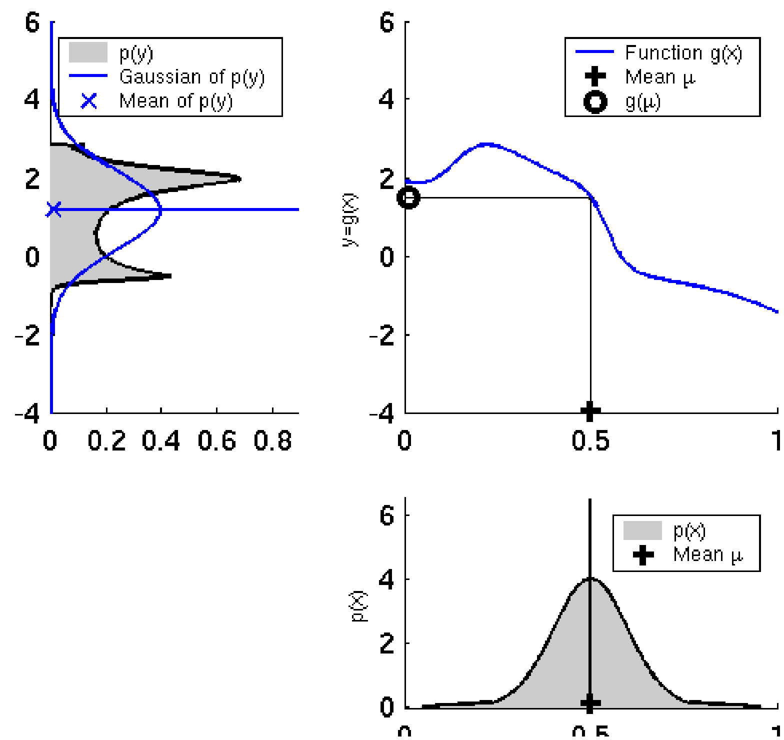
- **Second-Order Error Propagation**

Rarely used (complex expressions)

- **Monte-Carlo**

Non-parametric representation of uncertainties

1. Propagation of samples
2. Histogramming
3. Normalization



Derivations (1/4)

Result: Line Fit Cartesian Coordinates

(only for r, α more complicated...)

$$
\begin{aligned}
&\frac{\partial S}{\partial r} = 0\\
&\Leftrightarrow \frac{\partial}{\partial r}\{\sum \epsilon_i^2\} = \sum \frac{\partial}{\partial r}\{\epsilon_i^2\} = 2\sum \epsilon_i \frac{\partial}{\partial r}\{\epsilon_i\}\\
&\Leftrightarrow 2\sum (x_i cos\alpha + y_i sin\alpha - r)(-1) = 0\\
&\Leftrightarrow \sum (x_i cos\alpha + y_i sin\alpha - r) = 0\\
&\Leftrightarrow \sum x_i cos\alpha + \sum y_i sin\alpha - nr = 0\\
&\Leftrightarrow r = 1/n \sum x_i cos\alpha + 1/n \sum y_i sin\alpha\\
&\Leftrightarrow r = \bar{x} cos\alpha + \bar{y} sin\alpha
\end{aligned}
$$

Derivations (2/4)

Definitions

$$\mu = \mathrm{E}(X)$$
$$\mathrm{Var}(X) = \mathrm{E}((X-\mu)^2)$$
$$\mathrm{Cov}(X,Y) = \mathrm{E}((X-\mu)(Y-\nu))$$

Rules

$$\mathrm{E}(X+c) = \mathrm{E}(X) + c$$
$$\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$$
$$\mathrm{E}(aX) = a\mathrm{E}(X)$$

Result SISO

$$\mu_Y = f(\mu_X),$$

$$\sigma_Y = \left.\frac{\partial f}{\partial X}\right|_{X = \mu_X} \sigma_X .$$

Derivations (3/4)

Result MISO

$$
\begin{aligned}
\mu_Y = E[Y] &= E[a_0 + \sum_i a_i(X_i - \mu_i)] \\
&= E[a_0] + \sum_i E[a_i X_i] - E[a_i \mu_i] \\
&= a_0 + \sum_i a_i E[X_i] - a_i E[\mu_i] \\
&= a_0 + \sum_i a_i \mu_i - a_i \mu_i \\
&= a_0 \\
\mu_Y &= f(\mu_1, \mu_2, \ldots, \mu_n)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_Y^2 = E[(Y - \mu_Y)^2] &= E[(\sum_i a_i(X_i - \mu_i))^2] \\
&= E[\sum_i a_i(X_i - \mu_i) \sum_j a_j(X_j - \mu_j)] \\
&= E[\sum_i a_i^2 (X_i - \mu_i)^2 + \sum_{i \neq i}\sum a_i a_j (X_i - \mu_i)(X_j - \mu_j)] \\
&= \sum_i a_i^2 E[(X_i - \mu_i)^2] + \sum_{i \neq j}\sum a_i a_j E[(X_i - \mu_i)(X_i - \mu_j)] \\
&= \sum_i a_i^2 \sigma_i^2 + \sum_{i \neq j}\sum a_i a_j \sigma_{ij} \\
\sigma_Y^2 &= \sum_i \left(\frac{\partial f}{\partial X_i}\right)^2 \sigma_i^2 + \sum_{i \neq j}\sum \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial f}{\partial X_j}\right)\sigma_{ij}
\end{aligned}
$$

Derivations (4/4)

Result MIMO

$$\sigma_{YZ} = E[(Y-\mu_Y)(Z-\mu_Z)]$$

$$= E[Y \cdot Z] - E[Y]E[Z]$$

$$= E\left[\left(\mu_Y + \sum \frac{\partial f}{\partial X_i}[X_i - \mu_i]\right) \cdot \left(\mu_Z + \sum \frac{\partial g}{\partial X_i}[X_i - \mu_i]\right)\right] - \mu_Y \mu_Z$$

$$= E\left[\mu_Y \mu_Z + \mu_Z \sum \frac{\partial f}{\partial X_i}[X_i - \mu_i] + \mu_Y \sum \frac{\partial g}{\partial X_i}[X_i - \mu_i] + \sum \frac{\partial f}{\partial X_i}[X_i - \mu_i] \sum \frac{\partial g}{\partial X_i}[X_i - \mu_i]\right] - \mu_Y \mu_Z$$

$$= E[\mu_Y \mu_Z] + \mu_Z E\left[\sum \frac{\partial f}{\partial X_i} X_i - \sum \frac{\partial f}{\partial X_i} \mu_i\right] + \mu_Y E\left[\sum \frac{\partial g}{\partial X_i} X_i - \sum \frac{\partial g}{\partial X_i} \mu_i\right]$$

$$+ E\left[\sum \sum \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j}[X_i - \mu_i][X_j - \mu_j]\right] - \mu_Y \mu_Z$$

$$= \mu_Y \mu_Z + \mu_Z \sum \frac{\partial f}{\partial X_i} E[X_i] - \mu_Z \sum \frac{\partial f}{\partial X_i} E[\mu_i] + \mu_Y \sum \frac{\partial g}{\partial X_i} E[X_i] - \mu_Y \sum \frac{\partial g}{\partial X_i} E[\mu_i]$$

$$+ E\left[\sum \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_i}[X_i - \mu_i]^2 + \sum_{i \neq j} \sum \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j}[X_i - \mu_i][X_j - \mu_j]\right] - \mu_Y \mu_Z$$

$$= \sum \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_i} E[(X_i - \mu_i)^2] + \sum_{i \neq j} \sum \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} E[(X_i - \mu_i)(X_j - \mu_j)]$$

$$\sigma_{YZ} = \sum \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_i} \sigma_i^2 + \sum_{i \neq j} \sum \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} \sigma_{ij}^2$$