Introduction to Mobile Robotics

Probabilistic Robotics

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Probabilistic Robotics

Key idea:  
Explicit representation of uncertainty  
(using the calculus of probability theory)

- Perception = state estimation
- Action = utility optimization
Axioms of Probability Theory

Pr(A) denotes probability that proposition A is true.

- $0 \leq \Pr(A) \leq 1$

- $\Pr(True) = 1$ \hspace{1cm} $\Pr(False) = 0$

- $\Pr(A \lor B) = \Pr(A) + \Pr(B) - \Pr(A \land B)$
A Closer Look at Axiom 3

\[ \Pr(A \lor B) = \Pr(A) + \Pr(B) - \Pr(A \land B) \]
Using the Axioms

\[ \Pr( A \lor \neg A ) = \Pr(A) + \Pr(\neg A) - \Pr(A \land \neg A) \]
\[ \Pr(True) = \Pr(A) + \Pr(\neg A) - \Pr(False) \]
\[ 1 = \Pr(A) + \Pr(\neg A) - 0 \]
\[ \Pr(\neg A) = 1 - \Pr(A) \]
Discrete Random Variables

- $X$ denotes a random variable
- $X$ can take on a countable number of values in \{x_1, x_2, ..., x_n\}
- $P(X=x_i)$ or $P(x_i)$ is the probability that the random variable $X$ takes on value $x_i$
- $P(\cdot)$ is called probability mass function
- E.g. $P(Room) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$
Continuous Random Variables

- $X$ takes on values in the continuum.
- $p(X=x)$ or $p(x)$ is a probability density function.

$$Pr(x \in (a, b)) = \int_a^b p(x)dx$$

- E.g.
“Probability Sums up to One”

Discrete case

\[ \sum_x P(x) = 1 \]

Continuous case

\[ \int p(x) \, dx = 1 \]
Joint and Conditional Probability

- \( P(X=x \text{ and } Y=y) = P(x,y) \)

- If \( X \) and \( Y \) are independent then
  \[ P(x,y) = P(x) P(y) \]

- \( P(x | y) \) is the probability of \( x \) given \( y \)
  \[ P(x | y) = \frac{P(x,y)}{P(y)} \]
  \[ P(x,y) = P(x | y) P(y) \]

- If \( X \) and \( Y \) are independent then
  \[ P(x | y) = P(x) \]
Law of Total Probability

**Discrete case**

\[ P(x) = \sum_y P(x \mid y)P(y) \]

**Continuous case**

\[ p(x) = \int p(x \mid y)p(y) \, dy \]
Marginalization

Discrete case

\[ P(x) = \sum_{y} P(x, y) \]

Continuous case

\[ p(x) = \int p(x, y) \, dy \]
Bayes Formula

\[ P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x) \]

\[ \implies P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}} \]
Normalization

\[ P(x \mid y) = \frac{P(y \mid x) P(x)}{P(y)} = \eta \ P(y \mid x) P(x) \]

\[ \eta = P(y)^{-1} = \frac{1}{\sum_x P(y \mid x)P(x)} \]

**Algorithm:**

\[ \forall x : \text{aux}_{x \mid y} = P(y \mid x) \ P(x) \]

\[ \eta = \frac{1}{\sum_x \text{aux}_{x \mid y}} \]

\[ \forall x : P(x \mid y) = \eta \ \text{aux}_{x \mid y} \]
Bayes Rule
with Background Knowledge

\[ P(x \mid y, z) = \frac{P(y \mid x, z) \ P(x \mid z)}{P(y \mid z)} \]
Conditional Independence

\[ P(x, y \mid z) = P(x \mid z)P(y \mid z) \]

- Equivalent to \( P(x \mid z) = P(x \mid z, y) \)

and \( P(y \mid z) = P(y \mid z, x) \)

- But this does not necessarily mean

\[ P(x, y) = P(x)P(y) \]

(real independence)
Simple Example of State Estimation

- Suppose a robot obtains measurement $z$
- What is $P(\text{open}|z)$?
Causal vs. Diagnostic Reasoning

- $P(\text{open} | z)$ is diagnostic
- $P(z | \text{open})$ is causal
- Often causal knowledge is easier to obtain
- Bayes rule allows us to use causal knowledge:

$$P(\text{open} | z) = \frac{P(z | \text{open})P(\text{open})}{P(z)}$$

*count frequencies!*
Example

- $P(z|\text{open}) = 0.6 \quad P(z|\neg\text{open}) = 0.3$
- $P(\text{open}) = P(\neg\text{open}) = 0.5$

\[
P(\text{open} | z) = \frac{P(z | \text{open})P(\text{open})}{P(z | \text{open})p(\text{open}) + P(z | \neg\text{open})p(\neg\text{open})}
\]

\[
P(\text{open} | z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{0.3}{0.3 + 0.15} = 0.67
\]

- $z$ raises the probability that the door is open
Combining Evidence

- Suppose our robot obtains another observation $z_2$
- How can we integrate this new information?
- More generally, how can we estimate $P(x|z_1...z_n)$?
Recursive Bayesian Updating

\[ P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x, z_1, \ldots, z_{n-1}) P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})} \]

**Markov assumption:**

*z_n* is independent of *z_1, \ldots, z_{n-1}* if we know *x*

\[
P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x) P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}
\]

\[= \eta P(z_n \mid x) P(x \mid z_1, \ldots, z_{n-1})\]

\[= \eta_{1 \ldots n} \prod_{i=1}^{n} P(z_i \mid x) P(x)\]
Example: Second Measurement

- $P(z_2|\text{open}) = 0.5 \quad P(z_2|\neg\text{open}) = 0.6$
- $P(\text{open}|z_1) = \frac{2}{3}$

\[
P(\text{open} | z_2, z_1) = \frac{P(z_2 | \text{open}) \cdot P(\text{open} | z_1)}{P(z_2 | \text{open}) \cdot P(\text{open} | z_1) + P(z_2 | \neg\text{open}) \cdot P(\neg\text{open} | z_1)}
\]

\[
= \frac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{1}{2} \cdot \frac{2}{3} + \frac{3}{5} \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{5}} = \frac{\frac{3}{8}}{\frac{8}{15}} = \frac{5}{8} = 0.625
\]

- $z_2$ lowers the probability that the door is open
A Typical Pitfall

- Two possible locations $x_1$ and $x_2$
- $P(x_1)=0.99$
- $P(z|x_2)=0.09$  $P(z|x_1)=0.07$
Actions

- Often the world is **dynamic** since
  - actions carried out by the robot,
  - actions carried out by other agents,
  - or just the **time** passing by

- How can we **incorporate** such **actions**?
Typical Actions

- The robot **turns its wheels** to move
- The robot **uses its manipulator** to grasp an object
- Plants grow over **time**...

- Actions are **never carried out with absolute certainty**
- In contrast to measurements, **actions generally increase the uncertainty**
Modeling Actions

- To incorporate the outcome of an action $u$ into the current “belief”, we use the conditional pdf $P(x|u,x')$.

- This term specifies the pdf that executing $u$ changes the state from $x'$ to $x$. 
Example: Closing the door
State Transitions

\[ P(x|u,x') \] for \( u = \text{“close door”} \):

If the door is open, the action “close door” succeeds in 90% of all cases.
Integrating the Outcome of Actions

Continuous case:

\[ P(x \mid u) = \int P(x \mid u, x') P(x') \, dx' \]

Discrete case:

\[ P(x \mid u) = \sum P(x \mid u, x') P(x') \]
Example: The Resulting Belief

\[ P(\text{closed} \mid u) = \sum P(\text{closed} \mid u, x')P(x') \]
\[ = P(\text{closed} \mid u, \text{open})P(\text{open}) \]
\[ + P(\text{closed} \mid u, \text{closed})P(\text{closed}) \]
\[ = \frac{9}{10} \ast \frac{5}{8} + \frac{1}{8} \ast \frac{3}{16} = \frac{15}{16} \]

\[ P(\text{open} \mid u) = \sum P(\text{open} \mid u, x')P(x') \]
\[ = P(\text{open} \mid u, \text{open})P(\text{open}) \]
\[ + P(\text{open} \mid u, \text{closed})P(\text{closed}) \]
\[ = \frac{1}{10} \ast \frac{5}{8} + 0 \ast \frac{3}{16} = \frac{1}{16} \]
\[ = 1 - P(\text{closed} \mid u) \]
Bayes Filters: Framework

- **Given:**
  - Stream of observations $z$ and action data $u$:
    \[ d_t = \{ u_1, z_1 \ldots, u_t, z_t \} \]
  - Sensor model $P(z|x)$
  - Action model $P(x|u,x')$
  - Prior probability of the system state $P(x)$

- **Wanted:**
  - Estimate of the state $X$ of a dynamical system
  - The posterior of the state is also called **Belief**:
    \[ Bel(x_t) = P(x_t \mid u_1, z_1 \ldots, u_t, z_t) \]
Markov Assumption

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

\[
p(z_t \mid x_{0:t}, z_{1:t}, u_{1:t}) = p(z_t \mid x_t) \\
p(x_t \mid x_{1:t-1}, z_{1:t}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)
\]
Bayes Filters

\[ Bel(x_t) = P(x_t \mid u_1, z_1 \ldots, u_t, z_t) \]

Bayes

\[ = \eta \ P(z_t \mid x_t, u_1, z_1, \ldots, u_t) \ P(x_t \mid u_1, z_1, \ldots, u_t) \]

Markov

\[ = \eta \ P(z_t \mid x_t) \ P(x_t \mid u_1, z_1, \ldots, u_t) \]

Total prob.

\[ = \eta \ P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots, u_t, x_{t-1}) \ P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \, dx_{t-1} \]

Markov

\[ = \eta \ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \ P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \, dx_{t-1} \]

Markov

\[ = \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1}) \, dx_{t-1} \]

\[ = \eta \ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) \, dx_{t-1} \]
Algorithm Bayes_filter( Bel(x), d ):

1. \( \eta = 0 \)
2. If \( d \) is a perceptual data item \( z \) then
3. For all \( x \) do
4. \( Bel'(x) = P(z \mid x) Bel(x) \)
5. \( \eta = \eta + Bel'(x) \)
6. For all \( x \) do
7. \( Bel'(x) = \eta^{-1} Bel'(x) \)
8. Else if \( d \) is an action data item \( u \) then
9. For all \( x \) do
10. \( Bel'(x) = \int P(x \mid u, x') Bel(x') \, dx' \)
11. Return \( Bel'(x) \)
Bayes Filters are Familiar!

\[ Bel(x_t) = \eta \ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \ Bel(x_{t-1}) \ dx_{t-1} \]

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)
Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.