2.1. Pose in 2D

^A x_i.B = (x,y,th)
R = [cos(th) -sin(th); sin(th) cos(th)]

Example: ^A p = ^A R_B * ^B p

R \in SO(2) = Special Orthogonal Group: \(-\)R = R'  and  det ( R ) = +1

Also note that  R(-th) = (-1)R(th)  so rotation ccw is the inverse of rotation cw

Why only one parameter, th?  4 elements in a 2x2 matrix

Col (row) orthogonality ==> one constraint
Col (row) normality ==> two constraints

Therefore only 1 dof, so only one parameter is needed

Accounting for translation - eqs.(2.7 - 2.10)

Define SE(2) with elements known as homogeneous transformations

Notation: xi(x,y,th) \sim htform

Orange box. Give result of composition and inverse.

Introduce matlab toolbox function, se2(x,y,th)

Matlab example in fig 2.8

Introduce e2h (euclidean to homogeneous) and h2e

Introduce homtrans

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The pose of a rigid body is defined by three variables:

\[
\begin{bmatrix}
^A t_{B,x} \\
^A t_{B,y} \\
\theta
\end{bmatrix} = \begin{bmatrix}
^A t_{B}
\end{bmatrix}
\]

where  ^A t_B \in \mathbb{R}^2 = \text{space of 2-dimensional vectors}

\( a.k.a. \text{, Euclidean 2-space} \)

\( \theta \in S^1 = [0, 2\pi) \)

\( a.k.a. \text{, the circle of dimension 1} \)

Translate origin of \{B\} by \( t \) relative to \{A\}, i.e. ^A t_B

Rotate \{B\} about its new origin by \( \theta \).

Corke derives the 3-by-3 homogeneous transformation matrix that maps a point from \{B\} to \{A\}.
that maps a point from \{B\} to \{A\}.

\[
{^A T^*_B} = \begin{bmatrix}
{^A R_B} & {^A t_B} \\
0 & 0 & 1
\end{bmatrix}_{(3 \times 3)} = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) & (t_B)^x \\
\sin(\theta) & \cos(\theta) & (t_B)^y \\
0 & 0 & 1
\end{bmatrix}
\]

\[
{^A X_B} \quad {^A Y_B} \quad \text{origin of \{B\}}
\]

direction direction expresses in \{A\} and written in homogeneous form.

\[
{^A \tilde{p}} = {^A T^*_B} {^B \tilde{p}}
\]

where \( {^B \tilde{p}} = \begin{bmatrix} p \\ 1 \end{bmatrix} \) is the homogeneous form of \( p \).

\[
\begin{bmatrix}
{^A p} \\
1
\end{bmatrix} = \begin{bmatrix}
{^A R_B} & {^A t_B} \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
{^B p} \\
1
\end{bmatrix} = \begin{bmatrix}
{^A R_B} {^B p} + {^A t_B} \\
1
\end{bmatrix}
\]

\( {^A R_B} {^B p} \) expresses \( p \) in \{A\}.

\( {^A t_B} \) adds the displacement.

\( {^A R_B} \) is needed to write the \( {^B p} \) vector in \{A\} so it can be
added to the other vector in \( \{ A \} \), namely \( A^t \).

\[ \text{Special Euclidean Group } SE(2) = SO(2) \times \mathbb{R}^2 \]

\( \mathbb{R}^2 \) = Euclidean group = 2-dimension vector space.

"x" = Cartesian product or set product each element of \( SO(2) \) is combined with each element of \( \mathbb{R}(2) \)

\( R \in SO(2) \), SO stands for Special Orthogonal group of 2x2 matrices

\[ A_{2\times2} \in SO(2) \iff A^T = A^{-1} \Rightarrow A^T A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times2} \]

and \( \det(A) = 1 \)

Let \( a_1 \) and \( a_2 \) be columns of \( A \), ie, \( A = [a_1, a_2] \).

\[ A^T A^{-1} = I_{2\times2} \Rightarrow \|a_1\| = 1 \]

\[ \|a_2\| = 1 \]

\[ a_1^T a_2 = 0 \]

\( \left\{ \text{3 constraints} \right\} \)
A \((2 \times 2)\) matrix has 4 elements.
3 constraints \(\Rightarrow\) \(A\) has only 1 free variable!
\(\therefore\) Planar orientation can be represented with one variable!

\(\text{SO}(2)\) is a nice way to represent 2d orientation because it handles wrap-around

Combining an element of \(\text{SO}(2)\) and \(\mathbb{R}^2\) yields an element of \(\text{SE}(2)\).

Since \(\text{SO}(2)\) is 1-dimensional and \(\mathbb{R}^2\) is 2-dimensional, \(\text{SE}(2)\) is 3-dimensional.

Therefore 3 variables are needed to define a planar pose.

Each element of \(\text{SE}(2)\) is said to represent a rigid body displacement in the plane.

A couple special results:

\[
\Theta^A_{\Sigma_B} \sim (^A T_B)^T = \begin{bmatrix} ^A R_B & ^A t_B \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ^A R_B^T & -^B R_A^T \cdot ^A t_B \\ 0 & 1 \end{bmatrix}
\]
\[ ^{A}_{B} T_{C} \odot ^{A}_{C} T_{C} = ^{A}_{C} T_{C} = \begin{bmatrix} ^{A}_{B} R_{c} & ^{A}_{B} t_{c} \\ 0 &1 \end{bmatrix} \]

Given \( R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \), one can always find \( \theta \) by:

\[ \theta = \text{atan2}(r_{21}, r_{11}) \]

Rotations in the plane commute, i.e. \( R_{1} R_{2} = R_{2} R_{1} \)

Matlab functions:

\[ \text{se2}(x,y,\theta) \quad \text{se2}(1,2,\pi/6) = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 1 \\ 1/2 & \sqrt{3}/2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \]

axis
trplot2
hold on
plot_point
inv
e2h
h2e
homtrans