3.2_TimeVaryingCoordinateFrames

Wednesday, May 30, 2012
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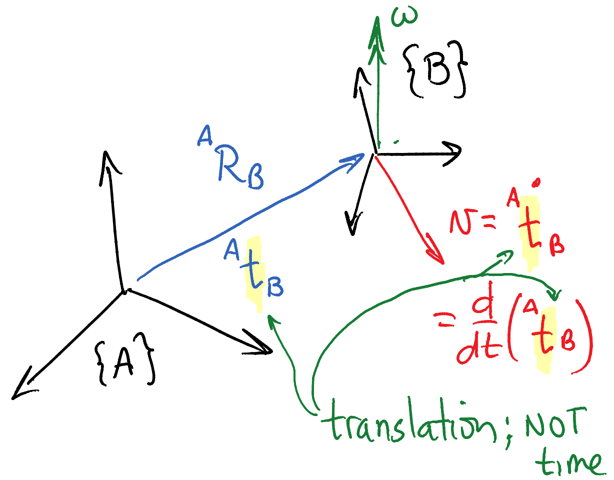
At each instant of time a time-varying coordinate frame has specific translational and rotational velocity vectors.



v = translation velocity of a reference point (usually the origin of {B}).

$v \in \mathbb{R}^3$

$^A v_B = \frac{d}{dt}({}^A t_B)$

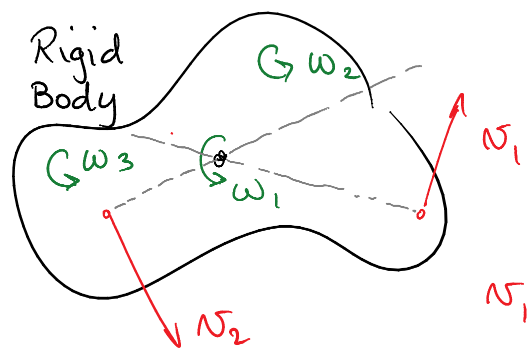
ω = rotational (or angular) velocity of a frame (or rigid body).

$\omega \in \mathbb{R}^3$

$^A\omega_B \neq \frac{d}{dt}({}^A R_B)$

$\neq \frac{d}{dt}(rpy)$

$\neq \frac{d}{dt}(\text{quaternion})$



$v_1 \neq v_2$

$\omega_1 = \omega_2 = \omega_3 = \dots$

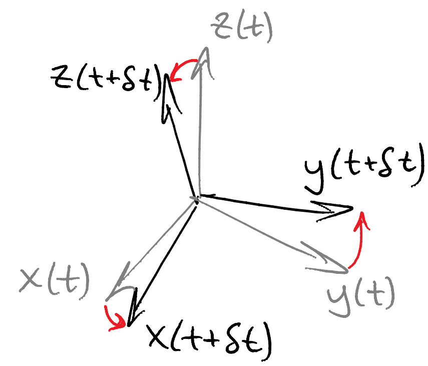
$\frac{\omega}{\|\omega\|} = \hat{\omega}$ defines an axis of rotation

$\|\omega\|$ defines the rate of rotation about the axis

3.2.1 Rotating Coordinate Frames

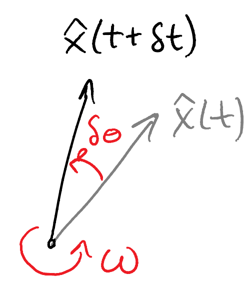
Goal: Define relationship for incremental changes in orientation and relate them to incremental changes in orientation parameters.

Time-varying frame $\Longrightarrow$



Consider only the change in R. (Origin is fixed.)

A view of $\quad \frac{d}{dt}(R(t)) = \dot{R}(t)$



$$\lim_{\delta t \to 0} \frac{\hat{x}(t+\delta t) - \hat{x}(t)}{\delta t} = \frac{d}{dt}(\hat{x}) = \dot{\hat{x}}$$

$$\omega\, \delta t = \delta \theta$$

$$\left.\begin{array}{l}\text{Since } \hat{x} \text{ has fixed length,} \\ \dot{\hat{x}} \text{ is due to orientation} \\ \text{change.}\end{array}\right\} \Rightarrow \frac{d}{dt}\hat{x} = \omega \times \hat{x}$$

Introduce cross product matrix $S(a)$, $a \in \mathbb{R}^3$

$$S(a) = \underset{(3\times3)}{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}}, \quad \text{note} \quad S^T(a) = -S(a)$$

Now $\dot{\hat{x}} = S(\omega)\,\hat{x}$.

Also $\dot{\hat{y}} = S(\omega)\,\hat{y}$, $\dot{\hat{z}} = S(\omega)\,\hat{z}$.

$$\Rightarrow \quad \dot{R} = \underset{(3\times3)}{\left[\dot{\hat{x}} \,\vdots\, \dot{\hat{y}} \,\vdots\, \dot{\hat{z}}\right]} = \left[S(\omega)\,\hat{x} \,\vdots\, S(\omega)\,\hat{y} \,\vdots\, S(\omega)\,\hat{z}\right]$$

$$\Rightarrow \quad \boxed{\dot{R}(t) = S(\omega)\,R(t)}$$

Toolbox

skew(ω) $\Rightarrow$ $S(\omega)$

vex($S(\omega)$) $\Rightarrow$ ω

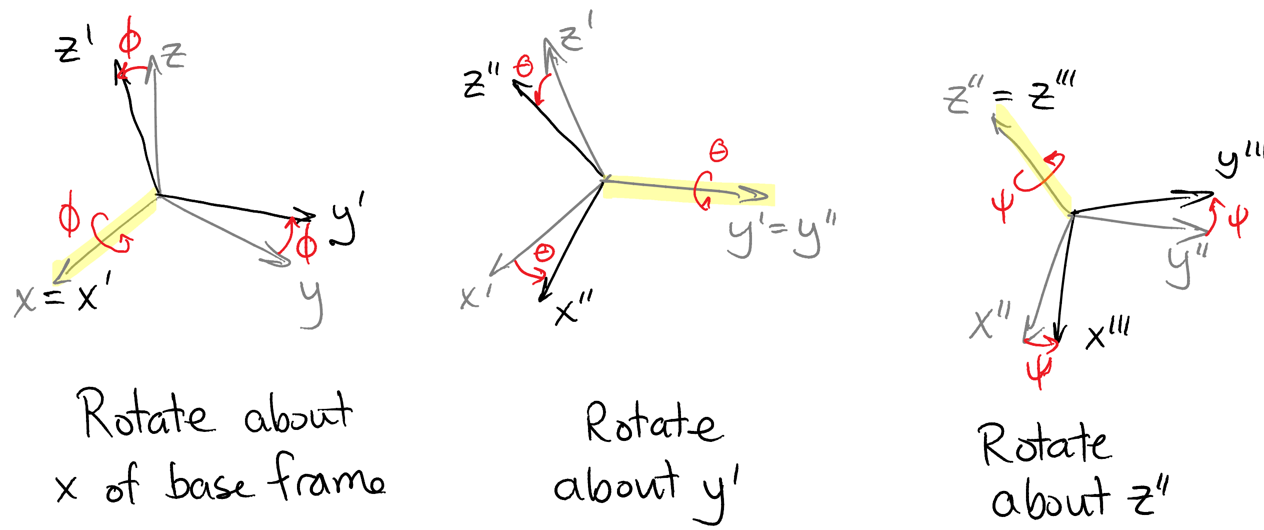
Next, a few specific rate relationships

(not in Corke's text)

Consider roll, pitch, yaw angles

$$R(rpy) = R_x(\theta_r)\, R_y(\theta_p)\, R_z(\theta_y)$$

to simplify notation use ϕ, θ, ψ for $\theta_r, \theta_p, \theta_y$



Now suppose we rotate the final frame with contributions $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ about each of the highlighted axes?

What would be the angular velocity of the final frame expressed in the base frame?

$$\omega = I \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + R_x(\phi) \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + R_x(\phi)\, R_y(\theta) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & S_\theta \\ 0 & C_\phi & -S_\phi C_\theta \\ 0 & S_\phi & C_\phi C_\theta \end{bmatrix}}_{B} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$\boxed{\omega = B\dot{\alpha}}, \quad \text{where } \dot{\alpha} = [\dot{\phi} \;\; \dot{\theta} \;\; \dot{\psi}]^T$$

What about the inverse relationship? Does B^{-1} exist?

Matlab's symbolic toolbox yields:

$$B^{-1} = \begin{bmatrix} 1 & S_\phi \tan(\theta) & -C_\phi \tan(\theta) \\ 0 & C_\phi & S_\phi \\ 0 & -S_\phi / C_\theta & C_\phi / C_\theta \end{bmatrix}$$

Singularity occurs when $\cos(\theta) = 0$

Now consider Euler angle representations.

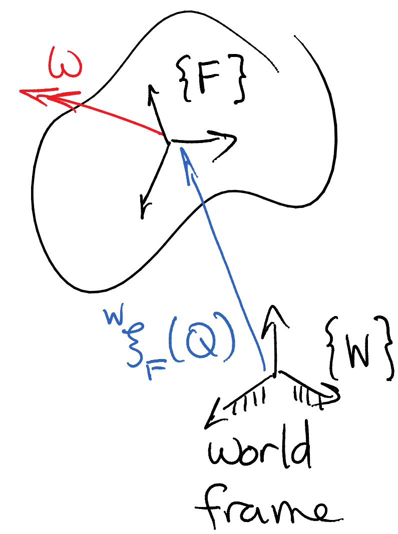
The exact same approach can be used. The Jacobian will have some singularities!

What about quaternions?

Let us represent $\mathring{q}$ as a vector of length 4.

$$\mathring{q} \sim Q = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \begin{array}{l} \text{scalar part} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{vector part} \end{array} \qquad \frac{d}{dt}Q = \frac{1}{2}B(Q)Q$$

It has been shown that $\frac{d}{dt}Q = \frac{1}{2}B(Q)\,{}^F\omega$



$$\text{where } B = \begin{bmatrix} -b & -c & -d \\ a & -d & c \\ d & a & -b \\ -c & b & a \end{bmatrix}$$

and ${}^F\omega$ is expressed in the body-fixed frame $\{F\}$

Incremental Relationships

$$\dot{R} \simeq \frac{R(t+\delta t) - R(t)}{\delta t}$$

$$\Rightarrow \quad R(t+\delta t) \simeq R(t) + \delta t\,\dot{R}(t) = R(t) + \delta t\, S(\omega) R(t)$$

$$\Rightarrow \quad \boxed{R(t+\delta t) \simeq \big(\delta t\, S(\omega) + I\big) R(t)}$$

$$\delta t\, S(\omega) + I = \begin{bmatrix} 1 & -\delta t\,\omega_z & \delta t\,\omega_y \\ \delta t\,\omega_z & 1 & -\delta t\,\omega_x \\ -\delta t\,\omega_y & \delta t\,\omega_x & 1 \end{bmatrix}$$

Note: Finite rotations do not commute, but infinitesimal rotations do! Composing infinitesimal rotations is equivalent to vector addition.

Compose two incremental rotations:

$$R(t+\delta t_1+\delta t_2) = \left(\delta t_2 S(\omega_2) + I\right)\left(\delta t_1 S(\omega_1) + I\right) R(t)$$

$$= \begin{bmatrix} 1 & -a_z & a_y \\ a_z & 1 & -a_x \\ -a_y & a_x & 1 \end{bmatrix} \begin{bmatrix} 1 & -b_z & b_y \\ b_z & 1 & -b_x \\ -b_y & b_x & 1 \end{bmatrix} R(t)$$

$$= \left[\begin{array}{c|c|c} 1 - a_z b_y - a_y b_z & & \\ a_z + b_z + a_x b_y & \text{SIMILAR} & \text{SIMILAR} \\ -(a_y + b_y) + a_x b_z & & \end{array}\right]$$

Assume: $a_i b_j \ll a_k$, $a_i b_j \ll b_k$, $i, j, k \in \{x, y, z\}$

Then $R(t+\delta t_1+\delta t_2) \simeq \begin{bmatrix} 1 & -(a_z+b_z) & a_y+b_y \\ a_z+b_z & 1 & -(a_x+b_x) \\ -(a_y+b_y) & a_x+b_x & 1 \end{bmatrix}$

Note that the result is independent of the order of incremental rotations

Mapping to incremental Cardan angles

$$R_1 = (\delta t\, S(\omega) + I) R_0$$

$$\Rightarrow R_1 R_0^T = (\delta t\, S(\omega) + I)$$

$$\Rightarrow R_1 R_0^T - I = \delta t\, S(\omega)$$

$$\text{vex}(R_1 R_0^T - I) = \text{vex}(\delta t\, S(\omega))$$

$$\delta\Theta = \delta t\, \omega$$

$$\delta t\, \omega \Rightarrow \text{rotx}(\delta t\, \omega_x) * \text{roty}(\delta t\, \omega_y) * \text{rotz}(\delta t\, \omega_z)$$

result is approx. independent of order

$\delta\Theta$ are interpretted as increments of Cardan angles, a.k.a., roll, pitch, yaw angles.

Note: when approximating R_1 by $(\delta t\, S(\omega) + I) R_0$,
the result is <u>NOT</u> exactly $\in SO(3)$!
You should normalize. (See 3.2.3)

3.2.2. Incremental Motion (Translation & Rotation)

Now include translation, t

Let $\delta = \Delta(T_0, T_1)$, where T_1 is very close to T_0

$$\delta = \begin{bmatrix} t_1 - t_2 \\ \mathrm{vex}(R_1 R_0^T - I) \end{bmatrix}$$

Toolbox: δ = tr2delta(T_0, T_1) $\qquad$ Δ = delta2tr(δ)

3.2.3 Inertial Navigation Systems

Orientation of a satellite, robot, etc can be estimated over time by integrating the output of a inertial measurement unit.

$$R(t+\delta t) = (\delta_t S(\omega) + I) R(t)$$ ⟸ Simplest update scheme. Runge-Kutta and other more accurate schemes are used.

δt is sampling rate

ω is output of sensor.

Make δt as small as possible to limit error, i.e. $R(t+\delta t) \notin SO(3)$

One Approach to Normalization of $R(t+\delta t)$

Let $R(t+\delta t) = [c_1' \; c_2' \; c_3']$, $R(t) = [c_1 \; c_2 \; c_3]$

Given ω, one can determine which c_i' changed the least by:

$$\omega^T R(t) = [d_1 \; d_2 \; d_3]$$

If d_i is $\min(d_1, d_2, d_3)$, then c_i changed least.

For argument sake, assume c_3 changed least.

Let new $c_3'' = c_3' / \|c_3'\|$

$$c_1'' = (c_2' \times c_3'') / (c_2' \times c_3'')$$

$$c_2'' = c_3'' \times c_1''$$

Corke normalizes all vectors last.

Using quaternions to increment orientation.

$$\frac{d}{dt}\mathring{q} = \frac{1}{2}\mathring{q}(\omega)\mathring{q} \quad \longleftarrow \quad \texttt{qd = q.dot(}\underbrace{\texttt{[1,2,3]}}_{\omega}\texttt{)}$$

$$\mathring{q}(t+\delta t) = \mathring{q}(t) + \delta t \frac{d}{dt}\mathring{q}(t) \quad \longleftarrow \quad \texttt{q2 = q1 + 0.001*qd}$$

$$\mathring{q}(t+\delta t)' = \mathring{q}(t+\delta t) / \|\mathring{q}(t+\delta t)\| \quad \longleftarrow \quad \texttt{q2 = q2.unit()}$$

The matrix-vector implementation of q.dot(ω) was given above. Recall:

$$\frac{d}{dt}(\mathring{q}) = \frac{1}{2} B(\mathring{q})\,{}^F\omega$$

where $\{F\}$ is the frame fixed in the rotating body.

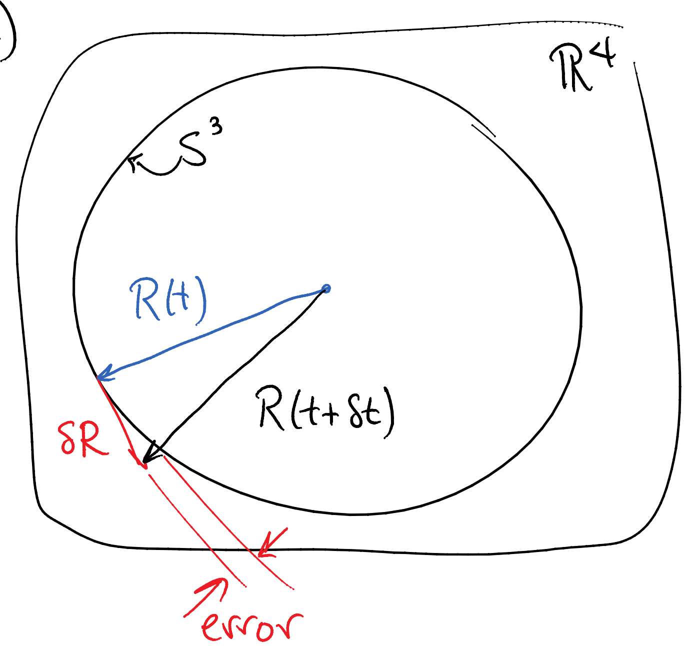
One last point about incremental changes to orientation.

SO(3) is a sphere, S^3, embedded in $\mathbb{R}^4$.

The simplest update (used above) is a vector tangent to S^3 added to a vector in S^3.

The result is not in S^3.

$\therefore$ Normalize!



More sophisticated updates such as Runge-Kutta updates tip δR so that it becomes a chord.