Joint Types:
- Revolute
- Prismatic

Arm Types:
- Revolute — all joints are revolute
- Scara — major links move in a horizontal plane. Usually used for vertical insertion tasks.

Task requirements:
- General tasks — end effector must have 6 dof.
  The set of end effector poses must be a 6D subset of \( \text{SE}(3) \).
- Vertical insertion tasks — end effector must cover a 4D subset of \( \text{SE}(2) \times \mathbb{I} \).
Pros - simpler, more accurate, faster
Cons - can't perform arbitrary tasks.

Workspace

The subset of $\mathbb{R}^n$ where the origin of the tool frame can be placed, where $n = 2$ or $3$.

$1R$ planar

\[ W = S' \cap I' \]
with joint limits.

$1P$-planar

\[ W = I' \]

$2R$-planar

with no joint limits,

\[ W = \text{disc with a hole} \]

with joint limits

\[ W = \text{disc} \]

$W$ of $3R$-planar robot is similar.
$W$ of all planar robots with small \# of $R$ and $P$ joints are easy to construct via enumeration.

**Dexterous Workspace, $W_0$**

**Def:** The subset of $W$ that can be reached with all end effector orientations.

Desirable for $W_0$ to be 2D for planar robots and 3D for spatial robots.

**Necessary conditions:**

- **Planar case:** At least one revolute joint w/o limits and at least two other joints to position the end effector.
- **Spatial case:** At least three revolute joints and at least three other joints to position the end effector.

Note: Joint limits will reduce volume of dexterous workspace.

**Solutions for boundary of $W_0$:**

- **3D:** Analytical solutions do not exist
  - Exact numerical solutions are very expensive.
- **2D:** Analytical solutions should exist
  - Exact numerical solutions should be easy to compute.
A topological perspective for 3R-planar robots

Pin the end effector to the ground and check if it can be rotated by \(2\pi\) radians.

**Theorem:**

Let \( L = \sum_{i=1}^{4} l_i \). Iff \( l_i + l_j \geq \frac{1}{2}L \ \forall \ i,j \in \{1,2,4\} \), then \( l_3 \) may rotate \(2\pi\) radian while pinned at \( p \).

If thm. is satisfied at \( p \), then it is satisfied at all points on the circle of radius \( l_4 \).

Now vary \( l_4 \) to "thicken" the dexterous workspace.

Experiment with some numbers.

Let \( l_1 = 5 \), \( l_2 = 4 \), \( l_3 = 1 \).

Also let \( \lambda_i \) be the \( i \)th longest link.

From the theorem, \( l_3 \) must be the shortest link and

\[
\lambda_2 + \lambda_3 > \frac{1}{2}L = \frac{1}{2}(l_1 + l_2 + l_3 + l_4)
\]
Clearly $0 \leq l_4 \leq 10$.

How big can $l_4$ be and still satisfy the conditions?

If $l_4 \geq 5$, then we require $l_1 + l_2 > \frac{1}{2}(l_1 + l_2 + l_3 + l_4)$

<table>
<thead>
<tr>
<th>$l_4$</th>
<th>$l_2 + l_3$</th>
<th>$l/2$</th>
</tr>
</thead>
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<tr>
<td>10</td>
<td>9</td>
<td>10</td>
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<td>9.5</td>
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<td>8</td>
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<td>6</td>
</tr>
<tr>
<td>1.9</td>
<td>5.9</td>
<td>5.95</td>
</tr>
</tbody>
</table>

Note: The theorem is valid for any number of revolute joints and prismatic joints can be incorporated by allowing the link lengths to vary.
Standard Denavit-Hartenberg parameters

This is a minimal representation
Only 4 parameters:
  3 are constant
  1 is variable

Frame assignment

- Identify joint axes.
  Label them \( z_j \), \( j=1,2,\ldots,N \)
- Identify common normal of \( z_{j-1} \) and \( z_j \), \( \forall j \)
  Label them \( x_j \)
- Define parameters
  \[ \theta_j = \text{angle about } z_{j-1} \text{ from } x_{j-1} \text{ to } x_j \]
  \[ d_j = \text{distance along } z_{j-1} \text{ from origin } \{j-1\} \text{ to origin } \{j\} \]
  \[ a_j = \text{distance along } x_j \text{ from } \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
  (i.e., the distance between \( z_{j-1} \) and \( z_j \))
  \[ \alpha_j = \text{angle about } x_j \text{ from } z_{j-1} \text{ to } z_j \]

Note: coordinate frames need not have origins inside the links.
\( \sigma_j = \begin{cases} 0 & \text{joint } j \text{ is revolute, and } \theta_j \text{ is the joint var.} \\ 1 & \text{joint } j \text{ is prismatic, and } d_j \text{ is the joint var.} \end{cases} \)

Relative pose:

\[
\begin{align*}
\hat{\gamma}_j A_j &= T_{Rz}(\theta_j) \ T_z(d_j) \ T_x(c_j) \ T_{Rx}(\alpha_j) \\
&= \begin{bmatrix}
C \theta & -S \alpha C \theta & S \alpha S \theta & a C \theta \\
S \theta & C \alpha C \theta & C \alpha S \theta & a S \theta \\
0 & -S \alpha & C \alpha & d \\
0 & 0 & 0 & 1
\end{bmatrix}_j
\end{align*}
\]

where all quantities have subscript, \( j \)

Note error in text. "\( a \)" is correct, not "\( \alpha \)".

Terminology:

Robot configuration: \( q \in \mathbb{R}^n \), \( q = [q_1, q_2, \ldots, q_n] \)

where \( q_i = \begin{cases} \theta_i & \text{if } \sigma_i = 0 \\ d_i & \text{if } \sigma_i = 1 \end{cases} \)

\( L = \text{link}([0, 0.1, 0.2, \pi/2, 0]) \)