7.3_InverseKinematics

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7:54 AM

Recall the Forward Kinematics Problem:

Determine ${}^{0}T_E$ given joint displacements, q.

Corke's notation $\quad \xi = \mathcal{K}(q) = {}^{0}T_E(q)$

solution is unique

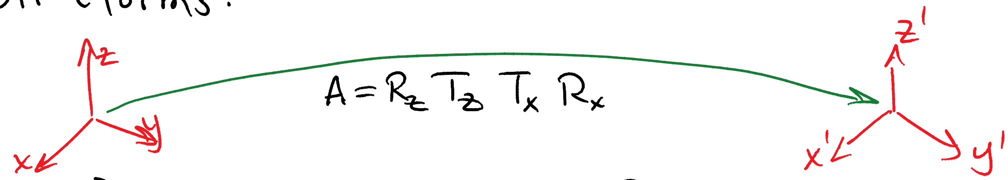
Forward Kinematic Map

Inverse Kinematics Problem:

Determine q given ${}^{0}T_E$, i.e. $q = \mathcal{K}^{-1}({}^{0}T_E)$.

When a solution exists, it is probably not unique.

Recall DH t'forms:



$$\{0\} \rightarrow R_z(q_1)T_z(d_1)T_x(a_1)R_x(\alpha_1) \rightarrow \{1\}$$

$$\{1\} \rightarrow R_z(q_2)T_z(d_2)T_x(a_2)R_x(\alpha_2) \rightarrow \{2\}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\{N\text{-}1\} \rightarrow R_z(q_N)\, T_z(d_N)\, T_x(a_N)\, R_x(\alpha_N) \rightarrow \{N\}$$

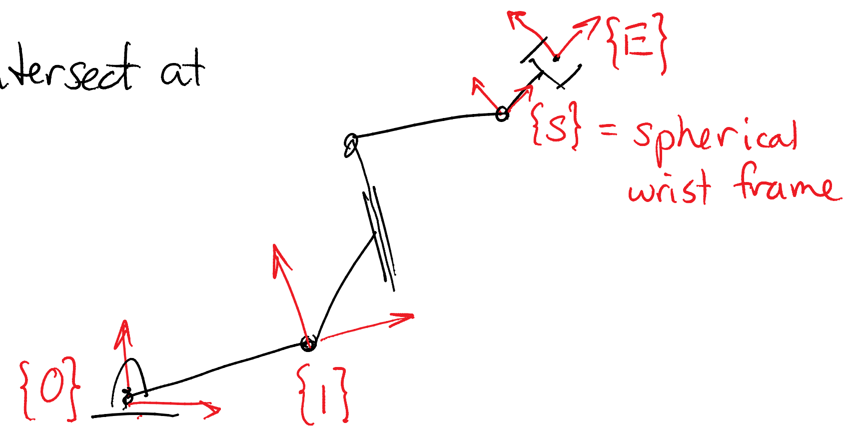
$${}^{0}A_1(q_1)\, {}^{1}A_2(q_2)\, {}^{2}A_3(q_3) \cdots {}^{N-2}A_{N-1}(q_{N-1})\, {}^{N-1}A_N(q_N) = {}^{0}T_N(q)$$

Given ${}^{0}T_{N,\text{target}}$, solve for $q \ni {}^{0}T_N(q) = {}^{0}T_{N,\text{target}}$

7.3.1 Closed-Form Solutions

Necessary condition – spherical wrist

$\Rightarrow$ last three axes intersect at a single point.

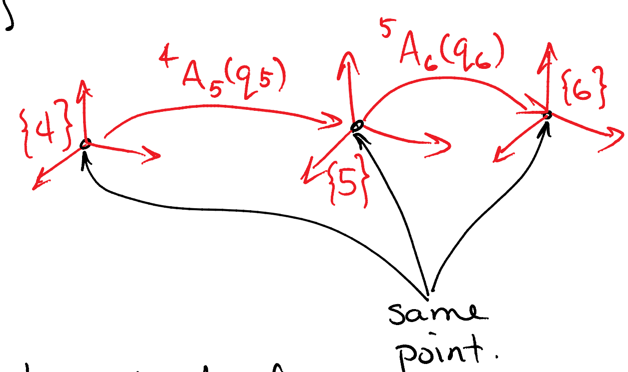


Spherical wrist $\Rightarrow$

- origins of $\{N-2\}$, $\{N-1\}$, & $\{N\}$ coincide,

  $a_j = d_j = 0, \quad j \in \{N-1, N\}$

$$^{N-2}A_{N-1} = R_z(q_{N-1}) \quad R_x(\alpha_{N-1})$$
$$^{N-1}A_N = R_z(q_N) \quad R_x(\alpha_N)$$



- position of wrist center is independent of

  q_{N-2}, q_{N-1}, & q_N.

Closed-form solution procedure

$$ {}^{W}T_0 \, {}^{0}T_N(q_N) \, {}^{N}T_E = \boxed{{}^{W}T_{E,\,Target}.} $$

— a given homogenous t'form matrix $\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

group all constant transformations

$$ \cancel{({}^{W}T_0)^{-1} \, {}^{W}T_0} \, {}^{0}T_N \, \cancel{{}^{N}T_E \, ({}^{N}T_E)^{-1}} = ({}^{W}T_0)^{-1} \, {}^{W}T_{E,Target} \, ({}^{N}T_E)^{-1} $$

$$ {}^{0}T_N(q) = {}^{W}T_0^{-1} \, {}^{W}T_{E,Target} \, {}^{N}T_E^{-1} $$

$$ {}^{0}T_N(q) = \underbrace{{}^{0}T_{N,\,Target}}_{\text{known}} $$

known

Next, translation and orientation subproblems

$$ {}^{0}T_N(q) = {}^{0}A_1(q_1) \, {}^{1}A_2(q_2) \cdots {}^{N-1}A_N(q_N) = {}^{0}T_{N,Target} $$

For simplicity, assume $N = 6$.

Expand A matrix product to reveal:

$$ \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{{}^{0}R_6(q_1,q_2,q_3,q_4,q_5,q_6)} & {}^{0}t_6(q_1,q_2,q_3) \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{{}^{0}R_{6,Target}} & {}^{0}t_{6,Target} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

Solve for q_1, q_2, q_3 using $\boxed{{}^0t_6(q_1,q_2,q_3) = {}^0t_{6,Target}}$

Then solve for q_4, q_5, q_6 using ${}^0R_6(q) = {}^0R_{6,Target}$

$$ {}^0R_3(q_1,q_2,q_3)\,{}^3R_6(q_4,q_5,q_6) = {}^0R_{6,Target} $$

$$ {}^3R_6(q_4,q_5,q_6) = {}^3R_0(q_1,q_2,q_3)\,{}^0R_{6,Target} $$

$$ \boxed{{}^3R_6(q_4,q_5,q_6) = {}^3R_{6,Target}} $$

Translation solution

$$ \underbrace{t_r(q_1,q_2,q_3)} = t_{r,Target} \quad ; \quad r \in \{x,y,z\} $$

trigonometric polynomials linear in $\sin(q_1)$ $\cos(q_1)$, $\sin(q_2)$, ...

Look for equations with solutions; e.g.

$$ a\cos(q_1) + b\sin(q_1) = c \quad , \quad a,b,c \in \mathbb{R} $$

then $\underline{q_1 = \text{atan2}(b,a) \pm \text{atan2}\left(\sqrt{a^2+b^2-c^2}, c\right)}$

Closed-form inverse kinematic solutions of 6-dof robot arms can be found in many text books.

Craig's book, "Intro. to Robotics" has the solution of the puma 560.

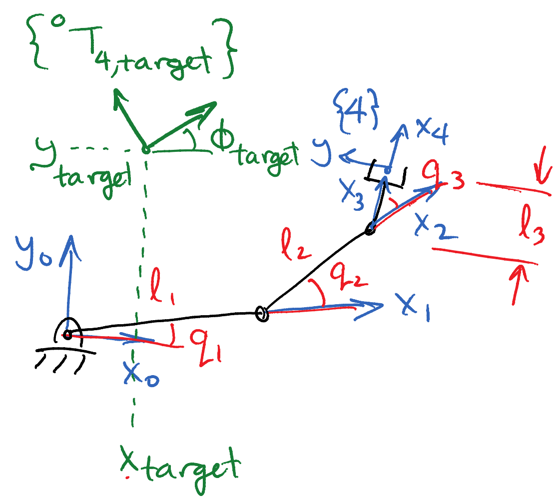
Uniqueness:

- Inv. kin. solutions are generally not unique.
- Most 6-joint robots have 8 inv. kin. solutions for each ${}^0T_{6,Target}$.
- If ${}^0T_{6,Target}$ corresponds to a singular config of the arm, then there will be an infinite # of solutions.

Planar Example: Inv. Kin. of 3R-Planar Manipulator

DH Parameters

i	θ_i	d_i	a_i	α_i
1	q_1	0	l_1	0
2	q_2	0	l_2	0
3	q_3	0	0	0
4	0	0	l_3	0



or let $a_3 = l_3$, then $\{4\} \equiv \{3\}$.

Inv. Kin. solution.

$$^0T_1(q_1)\,^1T_2(q_2)\,^2T_3(q_3)\,^3T_4 = \,^0T_4(q)$$

$$^0T_4(q) = \begin{bmatrix} c_{123} & -s_{123} & l_1c_1 + l_2c_{12} + l_3c_{123} \\ s_{123} & c_{123} & l_1s_1 + l_2s_{12} + l_3s_{123} \\ 0 & 0 & 1 \end{bmatrix}$$

where
$c_1 = \cos(q_1)$
$c_{12} = \cos(q_1+q_2)$
$c_{123} = \cos(q_1+q_2+q_3)$
s_1, s_{12}, s_{123} are defined similarly.

$$^0T_{4,\text{Target}} = \begin{bmatrix} c_\phi & -s_\phi & x_{target} \\ s_\phi & c_\phi & y_{target} \\ 0 & 0 & 1 \end{bmatrix}$$

Equate: $^0T_4(q) = \,^0T_{4,target}$

Nontrivial equations:

$$\phi_{target} = q_1 + q_2 + q_3 \qquad (1)$$

$$x_{target} = l_1c_1 + l_2c_{12} + l_3c_{123} \qquad (2)$$

$$y_{target} = l_1s_1 + l_2s_{12} + l_3s_{123} \qquad (3)$$

Drop "target" subscript for simplicity.

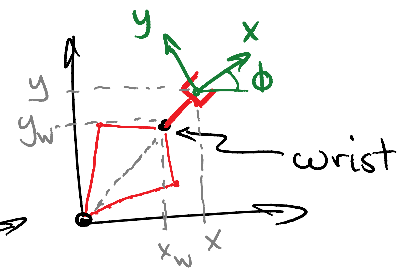
Note that $c_{123} = c_\phi$ and $s_{123} = s_\phi$ $\therefore$

$$x - l_3 c_\phi = l_1 c_1 + l_2 c_{12} \qquad (4)$$

$$y - l_3 s_\phi = l_1 s_1 + l_2 s_{12} \qquad (5)$$

Wrist position is fixed by choice of ${}^0T_{4,\text{Target}}$ $\Longrightarrow$

Two solution exist elbow-up & elbow-down



Square & add (4) & (5) to eliminate q_1

Let $x - l_3 c_\phi = x_w$, $y - l_3 s_\phi = y_w$

$$\left(x_w = l_1 c_1 + l_2 c_{12}\right)^2$$

$$+ \left(y_w = l_1 s_1 + l_2 s_{12}\right)^2$$

$$x_w^2 + y_w^2 = l_1^2 c_1^2 + 2 l_1 l_2 c_1 c_{12} + l_2^2 c_{12}^2$$

$$+ l_1^2 s_1^2 + 2 l_1 l_2 s_1 s_{12} + l_2^2 s_{12}^2$$

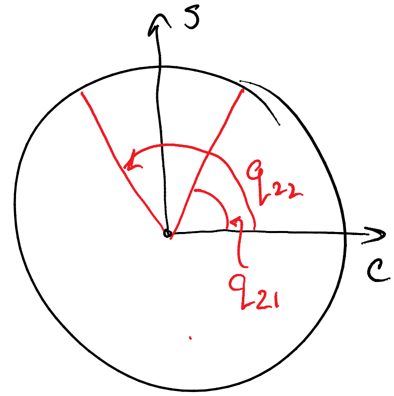
$$x_w^2 + y_w^2 = l_1^2 + 2 l_1 l_2 \cos(\cancel{q_1} + q_2 - \cancel{q_1}) + l_2^2$$

$$\boxed{\frac{x_w^2 + y_w^2 - l_1^2 - l_2^2}{2 l_1 l_2} = \cos(q_2)} \qquad \boxed{\sin(q_2) = \pm\sqrt{1 - \cos^2(q_2)}}$$

Two Solutions for q_2

$$q_{21} = \operatorname{atan2}\left(\sqrt{1-\cos^2(q_2)}, \cos(q_2)\right)$$

$$q_{22} = \operatorname{atan2}\left(-\sqrt{1-\cos^2(q_2)}, \cos(q_2)\right)$$



When are the solutions valid?

$$-1 \le \frac{x_w^2 + y_w^2 - \ell_1^2 - \ell_2^2}{2\ell_1\ell_2} \le 1 \Rightarrow \text{Workspace Boundary}$$

$$-2\ell_1\ell_2 \le x_w^2 + y_w^2 - \ell_1^2 - \ell_2^2 \le 2\ell_1\ell_2$$

$$\ell_1^2 + \ell_2^2 - 2\ell_1\ell_2 \le x_w^2 + y_w^2 \le \ell_1^2 + \ell_2^2 + 2\ell_1\ell_2$$

$$\boxed{(\ell_1 - \ell_2)^2 \le x_w^2 + y_w^2 \le (\ell_1 + \ell_2)^2} \Rightarrow$$

wrist can't be too far from base or too close to base.

Now q_2 is known and we know that $\phi = q_1 + q_2 + q_3$

How do we solve for q_1 ?

Expand (4) and (5) and rearrange ...

$$x_w = l_1 c_1 + l_2 (c_1 c_2 - s_1 s_2)$$

$$y_w = l_1 s_1 + l_2 (c_1 s_2 + s_1 c_2)$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} l_1 + l_2 c_2 & -l_2 s_2 \\ \underbrace{l_2 s_2}_{b(q_2)} & \underbrace{l_1 + l_2 c_2}_{a(q_2)} \end{bmatrix} \begin{bmatrix} c_1 \\ s_1 \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c_1 \\ s_1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ s_1 \end{bmatrix} = \underbrace{\frac{1}{a^2+b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}}_{\text{function of } q_2} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ s_1 \end{bmatrix} = \frac{1}{x_w^2 + y_w^2} \begin{bmatrix} a x_w + b y_w \\ -b x_w + a y_w \end{bmatrix} = \begin{bmatrix} A(q_2) \\ B(q_2) \end{bmatrix}$$

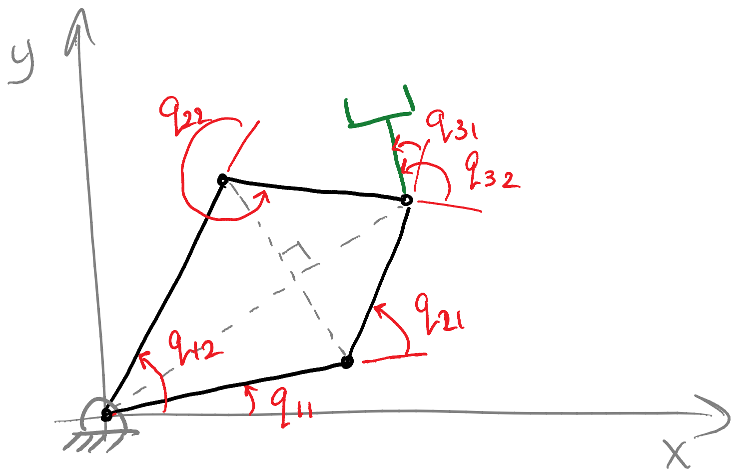
$$\boxed{\begin{aligned} q_{11} &= \operatorname{atan2}(B(q_{21}), A(q_{21})) \\ q_{12} &= \operatorname{atan2}(B(q_{22}), A(q_{22})) \end{aligned}}$$

A unique q_1 for each solution of q_2.

Finally

$$\boxed{\begin{aligned} q_{31} &= \phi - q_{11} - q_{21} \\ q_{32} &= \phi - q_{12} - q_{22} \end{aligned}}$$

Total # of solutions is 2.



Organized Approach to Solving Inv. Kinematics Problems

$$^0T_1(q_1)\,{}^1T_2(q_2)\,{}^2T_3(q_3)\cdots{}^{N-1}T_N(q_N) = {}^0T_{N,Target}$$

1.) Expand and note the nontrivial equations

2.) Look for equations that are easy to solve and attempt solution

3.) If not solved and some t'forms remain on the left side, premultiply by T^{-1}

$$^1T_2(q_2)\,{}^2T_3(q_3)\cdots{}^{N-1}T_N(q_N) = {}^0T_1^{-1}(q_1)\,{}^0T_{N,Target} = {}^1T_{N,Target}(q_1)$$

GO TO STEP 1.

Else, go to step 4

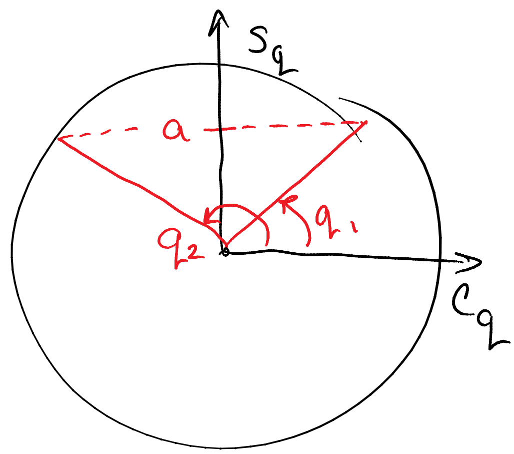
4.) STOP

For close form inv. kin. solution to exist, you need a good design and maybe some luck.

If possible give your robot a spherical wrist.

Typically occuring equations with known solutions

1.) $\sin(q) = a$

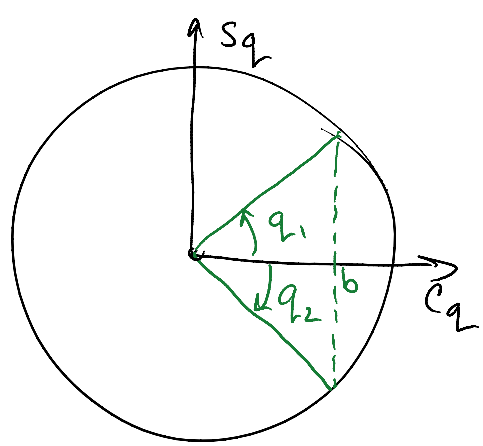


$q = \text{atan2}(a, \pm\sqrt{1-\cos^2(q)})$

$q_2 = \pi - q_1$

2. solutions

2.) $\cos(q) = b$



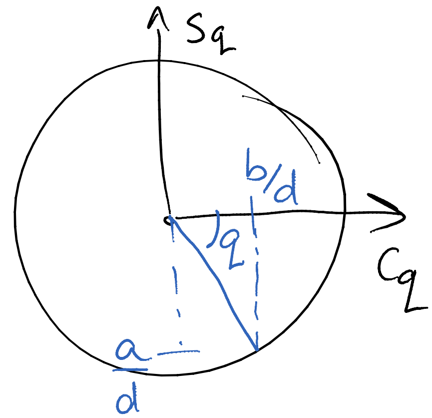
$q = \text{atan2}(\pm\sqrt{1-\cos^2(q)}, \cos(q))$

$q_2 = -q_1$

2 solutions

3. $d\sin(q) = a$

$d\cos(q) = b$



$$q = \text{atan2}(a/d, b/d)$$

1 solution

4. $a\cos(q) - b\sin(q) = 0$

$\Rightarrow \quad a\cos(q) = b\sin(q) \quad$ OR $\quad -a\cos(q) = -b\sin(q)$

$$q = \text{atan2}(\pm a, \pm b) \Rightarrow q_1 = q_2 + \pi$$

2 solutions

5. $\sin\theta \sin\phi = a$
$\cos\theta \sin\phi = b$, ϕ & θ unknown

Assume $\phi \neq n\pi$, $n \in \mathbb{Z}$ (the integers)

$$\left.\begin{aligned} \theta_1 &= \text{atan2}\left(\frac{a}{\sin\phi}, \frac{b}{\sin\phi}\right) \\ \theta_2 &= \theta_1 + \pi \end{aligned}\right\} \text{Two solutions}$$

What about ϕ ?

Square and add equations

$$\left.\begin{aligned} s_\theta^2 s_\phi^2 &= a^2 \\ c_\theta^2 s_\phi^2 &= b_2 \end{aligned}\right\} \Rightarrow s_\phi^2 = a^2+b^2$$

$\sin\phi = \pm\sqrt{a^2+b^2}$ $\Leftarrow$ two solutions for $\sin\phi$

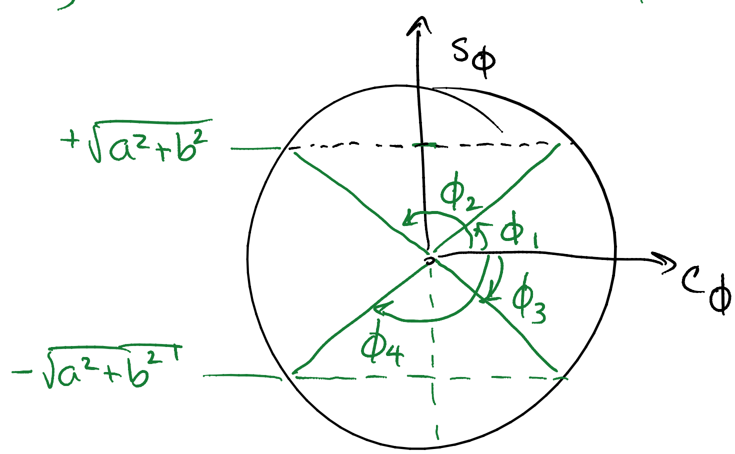
$\phi = \pm \sin^{-1}\left(\pm\sqrt{a^2+b^2}\right)$ $\Leftarrow$ 4 solutions for ϕ

$\phi_1 = \sin^{-1}\left(\sqrt{a^2+b^2}\right)$

$\phi_2 = \pi - \phi_1$

$\phi_3 = \phi_2 + \pi$

$\phi_4 = \phi_1 + \pi$



Note: There are still only 2 solutions for θ !

6. $\begin{cases} \sin\theta \sin\phi = a \\ \cos\theta \sin\phi = b \end{cases}$, θ & ϕ unknown

$\cos\phi = d$

$$\left.\begin{aligned} \sin\phi &= \pm\sqrt{a^2+b^2} \\ \cos\phi &= d \end{aligned}\right\} \Rightarrow \phi = \text{atan2}\left(\pm\sqrt{a^2+b^2}, d\right)$$

$$\theta = \text{atan2}(\pm a, \pm b) = \begin{cases} \text{atan2}\left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{(\cdot)}}\right) \\ \text{atan2}\left(\frac{-a}{\sqrt{}}, \frac{-b}{\sqrt{}}\right) \end{cases}$$

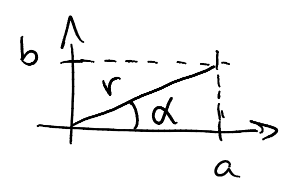
$$\phi_1 = \text{atan2}(\sqrt{}, d) \Rightarrow \theta_1 = \text{atan2}(a, b)$$

$$\phi_2 = -\phi_1 \Longrightarrow \theta_2 = \theta_1 + \pi$$

2 solutions

7. $a\cos(q) + b\sin(q) = c$

Let $\left.\begin{array}{l} a = r\cos(\alpha) \\ b = r\sin(\alpha) \end{array}\right\} \Rightarrow \begin{array}{l} r = \pm\sqrt{a^2+b^2} \\ \alpha = \text{atan2}(b/r, a/r) \end{array}$



Substitute:

$$r\cos(\alpha)\cos(q) + r\sin(\alpha)\sin(q) = c$$

Simplify:

$$\cos(q-\alpha) = c/r$$

Solve for q:

$$q-\alpha = \text{atan2}\left(\pm\sqrt{1-\frac{c^2}{r^2}}, \frac{c}{r}\right)$$

$$q-\alpha = \text{atan2}\left(\pm\sqrt{r^2-c^2}, c\right)$$

$$q = \operatorname{atan2}\left(\frac{b}{r}, \frac{a}{r}\right) + \operatorname{atan2}\left(\pm\sqrt{a^2+b^2-c^2}, c\right)$$

2 solutions

8. $$\left.\begin{aligned} a\cos(q) - b\sin(q) &= c \\ a\sin(q) + b\cos(q) &= d \end{aligned}\right\} \equiv \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} \cos(q) \\ \sin(q) \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$\begin{bmatrix} \cos(q) \\ \sin(q) \end{bmatrix} = \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}$$

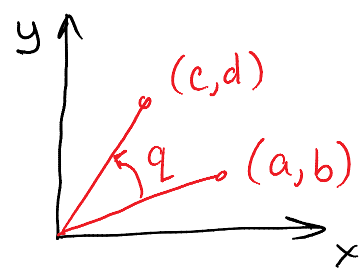
$$\cos(q) = (ac+bd)/(a^2+b^2)$$
$$\sin(q) = (ad-bc)/(a^2+b^2)$$

$$q = \operatorname{atan2}(ad-bc, ac+bd)$$

unique solution!

Physical interpretation of case 8.

Rotation of a vector $\longrightarrow$



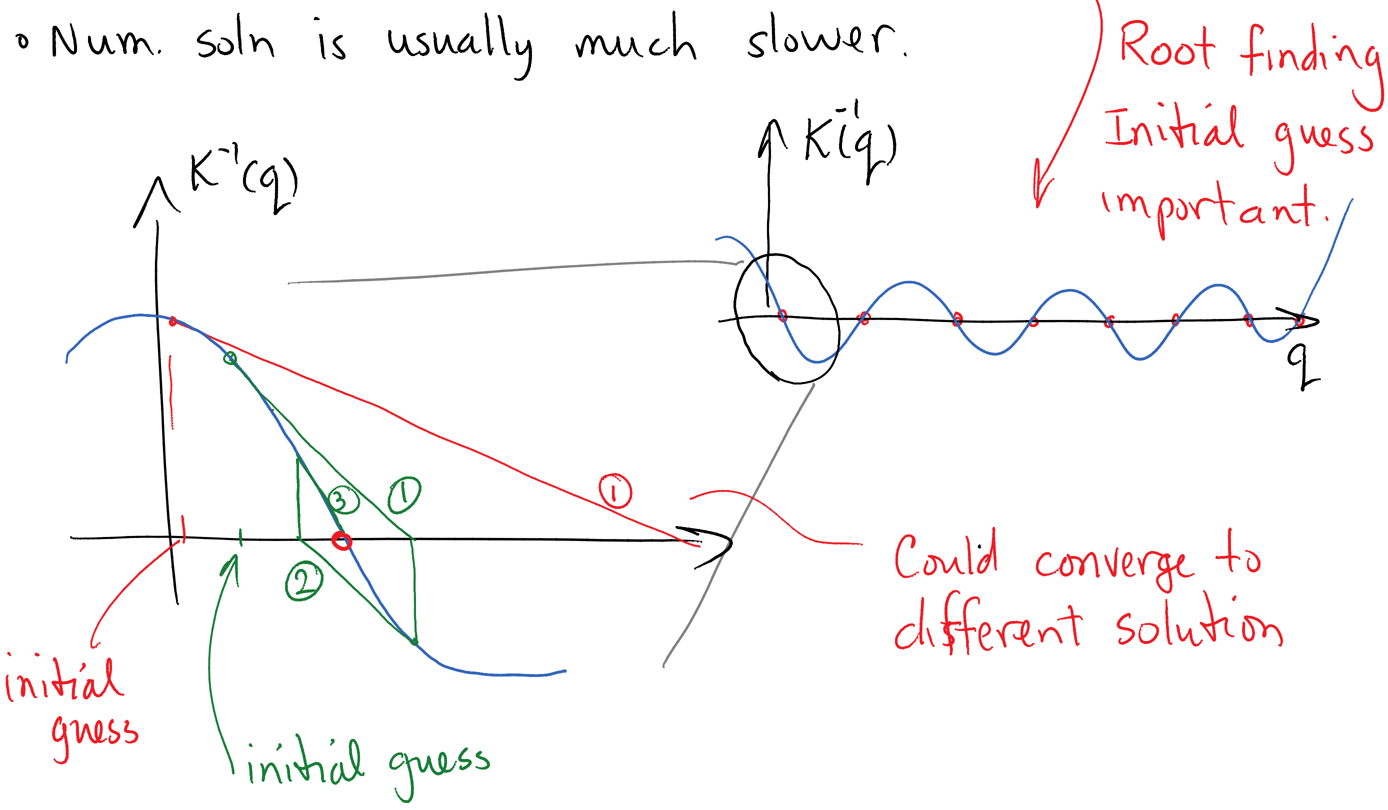
Note: If you square and the two equations, you get $\underline{a^2+b^2 = c^2+d^2}$.

7.3.2 Numerical Inverse Kinematic Solutions

The basics will be presented in Chapter 8.

Just a few comments here.

- Numerical solution is required when closed-form solutions do not exist
- Num. soln. does not give explicit control over the arm config., i.e., elbow-up, etc.
- Num. soln is usually much slower.



7.3.3. Under-Actuated Manipulators

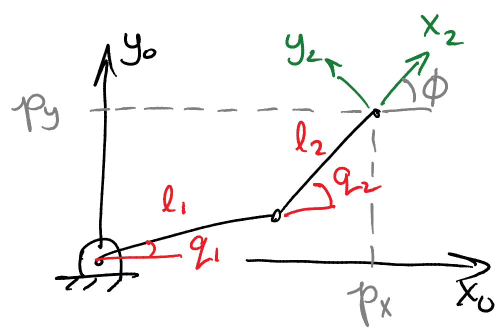
A robot is under-actuated if it has fewer controllable joints than the # of pose degrees of freedom (i.e. 6 in spatial systems, 3 in planar systems)

Special closed-form solutions can be derived.

For example, in the 2R-planar robot example, one could choose to solve equations (1) & (2) and ignore equation (3).

General numerical algorithm is implemented in the robotics toolbox, but it is not robust for all robots & starting guesses.

Example: 2R-Planar Manipulator



Inv. Kin. Problem

given: p_x, p_y, ϕ

find:

$$q \ni {}^0T_2(q) = \begin{bmatrix} c_\phi & -s_\phi & p_x \\ s_\phi & c_\phi & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$q \ni \ T_2(q) = \begin{bmatrix} s_\phi & c_\phi & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

There are only 2 joints, so solutions typically do not exist!

Choose what to satisfy

$$^0T_2(q) = \begin{bmatrix} c_{12} & -s_{12} & l_1 c_1 + l_2 c_{12} \\ s_{12} & c_{12} & l_1 s_1 + l_2 s_{12} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\phi & -s_\phi & p_x \\ s_\phi & c_\phi & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

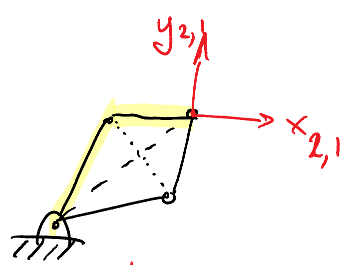
Nontrivial equations:

$$\phi = q_1 + q_2 \qquad (1)$$

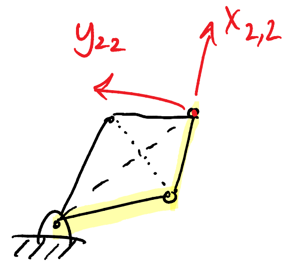
$$p_x = l_1 c_1 + l_2 c_{12} \qquad (2)$$

$$p_y = l_1 s_1 + l_2 s_{12} \qquad (3)$$

Choose to satisfy (1) & (2) $\Rightarrow$



$\phi_1 = q_{11} + q_{21}$



$\phi_2 = q_{21} + q_{22}$

Other options:

Choose to satisfy x & ϕ ?

7.3.4 Redundant Manipulators

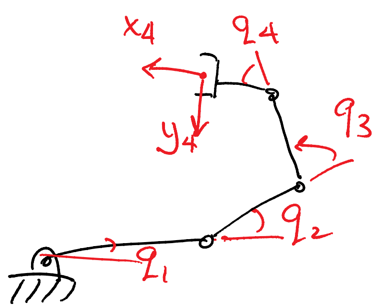
A manipulator is redundant if it has more joints than needed to achieve an arbitrary pose.

Closed-form solutions are possible if one joint displacement is chosen in advance.

e.g. Barret WAM.

Numerical solutions tend to find solutions "nearest" the initial guess.

Example: 4R-Planar Robot



Choose target p_x, p_y, ϕ

$$\phi = q_1 + q_2 + q_3 + q_4$$

$$p_x = l_1 c_1 + l_2 c_{12} + l_3 c_{123} + l_4 c_\phi$$

$$p_y = l_1 s_1 + l_2 s_{12} + l_3 s_{123} + l_4 s_\phi$$

Introduce another constraint!

x_4

Introduce another constraint!

For example, fix q_4 at a desired value.

Then robot reduces to 3R-planar

& solutions can be found.