8.1.1 Transforming Velocities between Coordinate Frames

\[ A_T^B = \begin{bmatrix} R_B & t_B \\ 0 & 1 \end{bmatrix} \]

Let \( \mathbf{v} = \begin{bmatrix} \mathbf{n} \\ \omega \end{bmatrix} \) be the spatial velocity of a point.

Choose origins as points of interest

\[ \mathbf{v}_A = \begin{bmatrix} \mathbf{n}_A \\ \omega_A \end{bmatrix}, \quad \mathbf{v}_B = \begin{bmatrix} \mathbf{n}_B \\ \omega_B \end{bmatrix} \]

\[ \begin{align*}
\mathbf{n}_B &= \mathbf{n}_A + \omega_A \times \mathbf{r} = \mathbf{I} \mathbf{n}_A - t_B \times \omega_A = \mathbf{I} \mathbf{n}_A - \mathbf{S}(t_B) \omega_A \\
\omega_B &= \omega_A \\
\Rightarrow \mathbf{v}_B &= \begin{bmatrix} \mathbf{n}_B \\ \omega_B \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\mathbf{S}(t_B) \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{n}_A \\ \omega_A \end{bmatrix} = \mathbf{J} \mathbf{v}_A
\end{align*} \]
Assume we know $\omega_A$ in $\{A\}$ and want to know $\omega_B$ in $\{B\}$.

$$A\omega_A = \omega_B,$$ but $$B\omega_B = R_A^T A\omega_B$$

$$R_A^T \left( A\stackrel{\cdot}{n}_B = A\stackrel{\cdot}{n}_A - S(A_t B) A\omega_A \right)$$

$$B\stackrel{\cdot}{n}_B = R_A^T A\stackrel{\cdot}{n}_A - R_A^T S(A_t B) A\omega_A$$

$$B\mathbf{v}_B = \begin{bmatrix} R_A & -R_A S(A_t B) & A\mathbf{v}_A \\ 0 & R_A^T & 0 \end{bmatrix}$$

Corke's result is achieved by recalling $R_A = R_B^T$, $S^T = -S$, and $(AB)^T = B^T A^T$

$$B\mathbf{v} = B\text{J}_A^T \mathbf{v}$$

Text also uses notation:

$$B\mathbf{v} = B\text{J}_A^T \mathbf{v}$$

$$B\mathbf{v} = J_v (A_t B^T) A\mathbf{v}$$

(8.3)

where $J_v (A_t B^T) = \begin{bmatrix} R_B^T & (S(A_t B) R_B^T) \end{bmatrix}$

(8.4)

Matlab function: \texttt{tr2jac} ($A\mathbf{t}_B$)
8.1.2. Jacobian in End Effector Frame

Recall in Corke's text,

\( \{W\} = \{O\} \quad \& \quad \{E\} = \{N\} \)

Eq.(8.2) \( \nu = J(q) \hat{q} \)

Here \( \nu = \omega_{E} \).

The Matlab expression is:

\[ \text{jacobi}(q) * \hat{q} = \omega_{E} \]

Now how do we map \( \omega_{E} \) to \( \nu_{E} \)?

\[ \text{jacobsn}(q) * \hat{q} = \nu_{E} \]

Confusion in text ... last equations in 8.1.2.

\[ \begin{align*}
\dot{\nu} & = J_{N}^{T_{N}} \nu_{N} \\
& = J_{N}(N_{T_{N}}) J_{N}(q) \hat{q} \\
& = J(q) \hat{q} \\
\Rightarrow \quad \dot{J}(q) & = J_{N}(N_{T}) J_{N}(q)
\end{align*} \]

But is this what is really meant?

This would compute the spatial velocity of the origin...
of \{0\} as if \{0\} were rigidly attached to \{N\}.

What I believe is intended is to express \(\nu_n\) in \{0\}, \(\nu_n \circ R_n \cdot 0 \circ R_n \nu_n \equiv \nu_n\)

8.1.3. Analytical Jacobian

We have \(T(q) = \begin{bmatrix} R(q) & t(q) \\ 0 & 1 \end{bmatrix}\)

Partition \(J\) as follows:

\[
\begin{bmatrix} N \\ \omega \end{bmatrix} = \begin{bmatrix} \frac{\partial q}{\partial q} \\ \frac{\partial \theta}{\partial q} \end{bmatrix} \dot{q}
\]

related to position of \(P\)
related to orientation of link containing \(P\).

\(N = \frac{d}{dt}(t(q)) = \frac{\partial t(q)}{\partial q} \dot{q}\)

\(J_p = \begin{bmatrix} \frac{\partial t_x}{\partial q_1} & \cdots & \frac{\partial t_x}{\partial q_n} \\ \frac{\partial t_y}{\partial q_1} & \cdots & \frac{\partial t_y}{\partial q_n} \\ \frac{\partial t_z}{\partial q_1} & \cdots & \frac{\partial t_z}{\partial q_n} \\ \frac{\partial \theta}{\partial q_1} & \cdots & \frac{\partial \theta}{\partial q_n} \end{bmatrix}\)
$J_\phi$ is more complicated.

We want $\omega = J_\phi(q) \dot{q}$

Recall $\dot{R}(q) = S(w) R(q)$

$\Rightarrow \dot{R}(q) R(q) = S(w) = \begin{bmatrix} 0 & -\omega_x & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$

Use $\dot{R} = \frac{\partial R}{\partial q_i} \dot{q}_i + \frac{\partial R}{\partial q_j} \dot{q}_j + \ldots + \frac{\partial R}{\partial q_n} \dot{q}_n$

Postmultiply by $R^T(q)$

Simplify to yield $\omega_x = \text{function of } q$ and linear in $\dot{q}$

$\omega_y = \ldots$

$\omega_z = \ldots$

Results can be arranged into $J_\phi \dot{q} = \omega$

Combining $J_p$ and $J_\phi$ yields:

$\nu = J(q) \dot{q}$, where the functional dependence on $q$ is known explicitly.
8.1.4. Jacobian Condition & Manipulability

\[ \nu = J(q) \dot{q} \]

row of zeros?

\[
\begin{bmatrix}
\vdots \\
N_2 \\
\vdots
\end{bmatrix} = \begin{bmatrix}
\text{Don't Care} \\
0 \\
\text{Don't Care}
\end{bmatrix} \begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2 \\
\vdots
\end{bmatrix}
\]

\[ \Rightarrow \text{corresp. velocity} = 0! \]

Column of zeros? \(\Rightarrow\) corresp. joint motion doesn't effect end-eff. vel.

Column of large values \(\Rightarrow\) corresp. joint has large effect.

The inverse relationship - \( \dot{q} = J^{-1} \nu \)

Column of large values \(\Rightarrow\) desired values of \( \nu \) can require large joint velocities
Determining singular configurations analytically.

Assume $J$ is $(6 \times 6)$.

Singular configs exist when $\det(J) = 0$.

These configs should be avoided when using Cartesian space moves.

One can identify singular configs analytically by solving $\det(J) = 0$.

Let $^{\circ}\mathcal{Q}_N = \{ q | \det(^{\circ}\mathcal{J}_N) = 0 \}$

$^{\circ}\mathcal{Q}_N = \{ q | \det(^{\circ}\mathcal{J}_N) = 0 \}$

Then $^{\circ}\mathcal{Q}_N = ^{\circ}\mathcal{Q}_N$.

The singular set is determined by the kinematic structure, not the frame of reference.
Example: 2R-Planar Manipulator.

\[ p_x = l_1 c_1 + l_2 c_{12} \]
\[ p_y = l_1 s_1 + l_2 s_{12} \]
\[ \phi = q_1 + q_2 \]

\[
\begin{bmatrix}
\dot{p}_x \\
\dot{p}_y \\
\dot{\phi}
\end{bmatrix} =
\begin{bmatrix}
-l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\
l_1 c_1 + l_2 c_{12} & l_2 c_{12}
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix}
\]

\[ \omega J_E (3 \times 2) \]

Every configuration is singular, because one cannot choose \( \dot{p}_x, \dot{p}_y, \dot{\phi} \) arbitrarily and determine corresponding \( \dot{q}_1, \dot{q}_2 \). 

Continue, ignoring \( \phi \).

\[
\begin{bmatrix}
\dot{p}_x \\
\dot{p}_y
\end{bmatrix}_{E_{org}} =
\begin{bmatrix}
-l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\
l_1 c_1 + l_2 c_{12} & l_2 c_{12}
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix}
\]

\[ w J_E (2 \times 2) \]
\[ \text{Det}(wJ_E) = J_{11}J_{22} - J_{12}J_{12} \quad \text{simplify} \quad l_1l_2s_2 \]

Singular configs are defined by \( l_1l_2s_2 = 0 \)

\[ s_2 = 0 \quad \text{iff} \quad \theta_2 = \pm n\pi, \quad n \in \mathbb{Z} \]

\[ \text{manipulability ellipse} \]

not singular

boundary singularity

\[ \begin{bmatrix}
- (l_1 + l_2) s_1 \\
( l_1 + l_2) c_1
\end{bmatrix}
\begin{bmatrix}
l_2 s_1 \\
l_2 c_1
\end{bmatrix} \]

Only motion tangent to workspace boundary is possible!

See Chap 8, Sec 1, demos Matlab code
Interpretation of columns of $J$ not same as eigenvectors of $JJ^T$.

Eigenvector of maximum eigenvalue gives the direction of greatest end-effector velocity, given that the joint velocities satisfy $\|q\| \leq 1$.

In this direction, the robot:

- has the lowest resolution for position control.
- is weakest for force application.
Another planar example: RPR manipulator

Fwd. Kin Model

<table>
<thead>
<tr>
<th>j</th>
<th>$\theta_j$</th>
<th>$d_j$</th>
<th>$a_j$</th>
<th>$\alpha_j$</th>
<th>$\tau_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$q_1$</td>
<td>0</td>
<td>$d$</td>
<td>$-\pi/2$</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$q_2$</td>
<td>0</td>
<td>$\pi/2$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$q_3$</td>
<td>0</td>
<td>$f$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$$\mathbf{T}_3(q) = \begin{bmatrix} c_{13} & -s_{13} & d_1 - q_2 s_1 + f c_{13} \\ s_{13} & c_{13} & d_1 + q_2 c_1 - f s_{13} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{J}_3 = \begin{bmatrix} -d_1 - q_2 c_1 - f s_{13} & -s_1 & -f s_{13} \\ d_1 - q_2 s_1 - f c_{13} & c_1 & -f c_{13} \\ 1 & 0 & 1 \end{bmatrix}$$

Length of velocity vector is:
\[\| r_1(q) \| \dot{q}_1 \]
\[\| r_3(q_3) \| \dot{q}_3 \]
Let \( \|r_3\| = 1 \). What direction will yield fastest motion for \( \|q_2\| = 1 \)?

What about singularities?

Matlab's symbolic toolbox gives \( \det(J) = -q_2 \)

\( q_2 = 0 \Rightarrow \) revolute joint lies along the line of the first link.

This is an interior singularity!

Look at \( J \)

\[
\begin{bmatrix}
-ds_1 -fs_{13} & -s_1 & -fs_{13} \\
-dc_1 -fe_{13} & c_1 & -fc_{13} \\
1 & 0 & 1
\end{bmatrix}
\]

The columns are linearly dependent!

\( \text{col1} = d \times \text{col2} + \text{col3} \).

Rank \((J) = 2 \) \( \Rightarrow \) only velocities in a 2D subspace of \((\nu_x, \nu_y, \omega_z)\) can be achieved.
What velocities are impossible?

null(J) is 1 dimensional.

What motion causes no end-effector motion?

Are there any boundary singularities? No!

Kuka Jacobian?