Foundations of Artificial Intelligence

5. Constraint Satisfaction Problems
CSPs as Search Problems, Solving CSPs, Problem Structure

Wolfram Burgard, Bernhard Nebel, and Martin Riedmiller

Albert-Ludwigs-Universität Freiburg

Mai 11, 2012
Contents

1 What are CSPs?
2 Backtracking Search for CSPs
3 CSP Heuristics
4 Constraint Propagation
5 Problem Structure
A Constraint Satisfaction Problems (CSP) consists of:
- a set of variables \( \{X_1, X_2, \ldots, X_n\} \) to which
- values \( \{d_1, d_2, \ldots, d_k\} \) can be assigned
- such that a set of constraints over the variables is respected

A CSP is solved by a variable assignment that satisfies all given constraints.

In CSPs, states are explicitly represented as variable assignments. CSP search algorithms take advantage of this structure.

The main idea is to exploit the constraints to eliminate large portions of search space.

*Formal representation language* with associated general inference algorithms.
Example: Map-Coloring

- **Variables:** $WA, NT, SA, Q, NSW, V, T$
- **Values:** \{red, green, blue\}
- **Constraints:** adjacent regions must have different colors, e.g., $NSW \neq V$
Australian Capital Territory (ACT) and Canberra (inside NSW)

View of the Australian National University and Telstra Tower
Solution assignment:

- \{ WA = \text{red}, \ NT = \text{green}, \ Q = \text{red}, \ NSW = \text{green}, \ V = \text{red}, \ SA = \text{blue}, \ T = \text{green} \}

- Perhaps in addition \( ACT = \text{blue} \)
a constraint graph can be used to visualize binary constraints
for higher order constraints, hyper-graph representations might be used
Nodes = variables, arcs = constraints

Note: Our problem is 3-colorability for a planar graph
Variations

- Binary, ternary, or even higher arity (e.g., ALLDIFFERENT)

- **Finite** domains ($d$ values) $\rightarrow d^n$ possible variable assignments

- **Infinite** domains (reals, integers)
  - *linear constraints*: solvable (in P if real)
  - *nonlinear constraints*: unsolvable
Applications

- Timetabling (classes, rooms, times)
- Configuration (hardware, cars, ...)
- Spreadsheets
- Scheduling
- Floor planning
- Frequency assignments
- Sudoku
- ...
Backtracking Search over Assignments

- Assign values to variables step by step (order does not matter)
- Consider only one variable per search node!
- **DFS with single-variable assignments is called backtracking search**
- Can solve $n$-queens for $n \approx 25$
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
    return BACKTRACK({}, csp)

function BACKTRACK(assignment, csp) returns a solution, or failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment then
            add \{var = value\} to assignment
        inferences ← INERENCE(csp, var, value)
        if inferences ≠ failure then
            add inferences to assignment
            result ← BACKTRACK(assignment, csp)
            if result ≠ failure then
                return result
            remove \{var = value\} and inferences from assignment
    return failure
Example (2)
Improving Efficiency:
CSP Heuristics & Pruning Techniques

- **Variable ordering**: Which one to assign first?
- **Value ordering**: Which value to try first?
- Try to **detect failures** early on
- Try to exploit **problem structure**

→ **Note**: all this is not problem-specific!
Variable Ordering:
Most constrained first

- Most constrained variable:
  - choose the variable with the fewest remaining legal values
  → reduces branching factor!
Variable Ordering: Most Constraining Variable First

- Break ties among variables with the same number of remaining legal values:
  - choose variable with the most constraints on remaining unassigned variables
- reduces branching factor in the next steps
Value Ordering:
Least Constraining Value First

- Given a variable,
  - choose first a value that rules out the fewest values in the remaining unassigned variables
  
→ We want to find an assignment that satisfies the constraints (of course, does not help if unsat.)
Rule out Failures early on: Forward Checking

- Whenever a value is assigned to a variable, values that are now illegal for other variables are removed.
- Implements what the ordering heuristics implicitly compute.
- $WA = \text{red}$, then $NT$ cannot become $\text{red}$.
- If all values are removed for one variable, we can stop!
Forward Checking (1)

- Keep track of remaining values
- Stop if all have been removed
Forward Checking (2)

- Keep track of remaining values
- Stop if all have been removed
Forward Checking (3)

- Keep track of remaining values
- Stop if all have been removed
Forward Checking (4)

- Keep track of remaining values
- Stop if all have been removed
Forward Checking: Sometimes it Misses Something

- Forward Checking propagates information from assigned to unassigned variables.
- However, there is no propagation between unassigned variables.

![Diagram showing Forward Checking process with assignment of states to territories in Australia.]

<table>
<thead>
<tr>
<th>WA</th>
<th>NT</th>
<th>Q</th>
<th>NSW</th>
<th>V</th>
<th>SA</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A directed arc $X \rightarrow Y$ is “consistent” iff
- for every value $x$ of $X$, there exists a value $y$ of $Y$, such that $(x, y)$ satisfies the constraint between $X$ and $Y$

Remove values from the domain of $X$ to enforce arc-consistency

Arc consistency detects failures earlier

Can be used as preprocessing technique or as a propagation step during backtracking
Arc Consistency Example
function AC-3(csp) returns false if an inconsistency is found and true otherwise

inputs: csp, a binary CSP with components \((X, D, C)\)

local variables: queue, a queue of arcs, initially all the arcs in csp

while queue is not empty do
  \((X_i, X_j)\) ← REMOVE-FIRST(queue)
  if REVISE(csp, X_i, X_j) then
    if size of \(D_i = 0\) then return false
    for each \(X_k\) in \(X_i\).NEIGHBORS - \(\{X_j\}\) do
      add \((X_k, X_i)\) to queue
  return true

function REVISE(csp, X_i, X_j) returns true iff we revise the domain of \(X_i\)

revised ← false
for each \(x\) in \(D_i\) do
  if no value \(y\) in \(D_j\) allows \((x,y)\) to satisfy the constraint between \(X_i\) and \(X_j\) then
    delete \(x\) from \(D_i\)
    revised ← true
return revised
AC3 runs in $O(d^3n^2)$ time, with $n$ being the number of nodes and $d$ being the maximal number of elements in a domain.

Of course, AC3 does not detect all inconsistencies (which is an NP-hard problem).
CSP has two independent components
- Identifiable as connected components of constraint graph
- Can reduce the search space dramatically
Problem Structure (2): Tree-structured CSPs

- If the CSP graph is a tree, then it can be solved in $O(nd^2)$
- General CSPs need in the worst case $O(d^n)$

*Idea:* Pick root, order nodes, apply arc consistency from leaves to root, and assign values starting at root
Problem Structure (2): Tree-structured CSPs

- Pick any variable as root; choose an ordering such that each variable appears after its parent in the tree.
- Apply arc-consistency to \((X_i, X_k)\), when \(X_i\) is the parent of \(X_k\), for all \(k = n\) down to 2. (any tree with \(n\) nodes has \(n - 1\) arcs, per arc \(d^2\) comparisons are needed: \(O(n \cdot d^2)\))
- Now one can start at \(X_1\) assigning values from the remaining domains without creating any conflict in one sweep through the tree!
- Algorithm linear in \(n\)
Problem Structure (3): Almost Tree-structured

Idea: Reduce the graph structure to a tree by fixing values in a reasonably chosen subset

Instantiate a variable and prune values in neighboring variables is called **Conditioning**
Problem Structure (4): Almost Tree-structured

Algorithm Cutset Conditioning:

1. Choose a subset $S$ of the CSPs variables such that the constraint graph becomes a tree after removal of $S$. $S$ is called a cycle cutset.

2. For each possible assignment of variables in $S$ that satisfies all constraints on $S$
   - remove from the domains of the remaining variables any values that are inconsistent with the assignments for $S$, and
   - if the remaining CSP has a solution, return it together with the assignment for $S$

Note: Finding the smallest cycle cutset is NP hard, but several efficient approximation algorithms are known.
Another Method: Tree Decomposition (1)

- Decompose problem into a set of connected sub-problems, where two sub-problems are connected when they share a constraint
- Solve sub-problems independently and combine solutions
Another Method: 
Tree Decomposition (2)

- A **tree decomposition** must satisfy the following conditions:
  - **Every variable** of the original problem appears in at least one sub-problem
  - **Every constraint** appears in at least one sub-problem
  - If a variable appears in two sub-problems, it must appear in **all sub-problems on the path** between the two sub-problems
  - The connections form a **tree**
Another Method: Tree Decomposition (3)

- Consider sub-problems as new **mega-variables**, which have values defined by the solutions to the sub-problems.
- Use technique for **tree-structured CSP** to find an overall solution (constraint is to have identical values for the same variable).

\[
\begin{align*}
\{WA=\text{red}, NT=\text{green}, SA=\text{blue}\} \\
\{WA=\text{red}, NT=\text{blue}, SA=\text{green}\} \\
\{WA=\text{blue}, NT=\text{green}, SA=\text{red}\} \\
\vdots
\end{align*}
\]
The aim is to make the subproblems as small as possible. **Tree width** $w$ of a tree decomposition is the size of largest sub-problem minus 1.

**Tree width of a graph** is minimal tree width over all possible tree decompositions.

If a graph has tree width $w$ and we know a tree decomposition with that width, we can solve the problem in $O(nd^{w+1})$.

Unfortunately, finding a tree decomposition with minimal tree width is **NP-hard**. However, there are heuristic methods that work well in practice.
CSPs are a special kind of search problem:
- states are value assignments
- goal test is defined by constraints

Backtracking = DFS with one variable assigned per node. Other intelligent backtracking techniques possible.

Variable/value ordering heuristics can help dramatically.

Constraint propagation prunes the search space.

Path-consistency is a constraint propagation technique for triples of variables.

Tree structure of CSP graph simplifies problem significantly.

Cutset conditioning and tree decomposition are two ways to transform part of the problem into a tree.

CSPs can also be solved using local search.