Exercise 1.1 (Graphs)

Let $\mathcal{G} := (G, E)$ be an undirected graph. We define the degree of a vertex $g \in G$ to be the number of edges incident to $g$. We say that $\mathcal{G}$ is $k$-regular ($k \geq 0$) if each vertex $g \in G$ has degree $k$, or, equivalently, if every vertex is directly connected by an edge to exactly $k$ other vertices.

Prove that for every integer $k \geq 2$ there exists a $k$-regular graph $\mathcal{G}_k := (G_k, E_k)$ so that $|G_k| = 2k$ and diameter of $\mathcal{G}_k$ is 2.

We recall that the diameter is the longest shortest path between any two vertices of the graph.

Hint: you can show the statement above either by construction or by induction.

Exercise 1.2 (DFA)

Consider the following two DFAs (deterministic finite automata) with $\Sigma = \{0, 1\}$:

(a) What languages ($L_1$ and $L_2$) do these two automata individually recognize?

(b) Give the formal definition for $M_1$.

(c) Show that $L_1 \cup L_2$ is also a regular language, by constructing one DFA. Please hand in a high quality diagram.
Exercise 1.3 (DFA)

(a) Construct a DFA that recognizes the language $L$ with an alphabet $\Sigma = \{0, 1\}$, where
$L = \{w \mid w$ has both an even number of 0's and an even number of 1's$\}$

(b) Give the state diagram for a DFA accepting the language
$L = \{w \mid w$ starts with 1 and contains 10 or starts with 0 and contains the 01$\}$
The alphabet is $\Sigma = \{0, 1\}$.

Exercise 1.4 (Regular Language)

In this exercise, we want to prove that regular languages are closed under intersection and under complement. The intersection of two languages is defined as $L_1 \cap L_2$. The complement of a language is defined as the set of all words in $\Sigma^*$ which are not in $L$, i.e. $\bar{L} = \Sigma^* \setminus L$ ($\Sigma^*$ is the set of all words/strings over $\Sigma$).

Let $L$ and $L'$ be regular languages that are recognized by DFAs $M = (Q, \Sigma, \delta, q_0, F)$ and $M' = (Q', \Sigma', \delta', q'_0, F')$, respectively.

(a) Show that the regular languages are closed under intersection, i.e. give a finite automaton that recognizes $L \cap L'$.

(b) Show that the regular languages are closed under complement, i.e. give a finite automaton that recognizes $\bar{L}$. 