Exercise 8.1 (Runtime)
You have implemented an algorithm that needs exactly \( f(n) \) steps to terminate, where \( n \) is the size of the input. Assume that on your machine each step takes \( 1 \mu s \).
For which maximal input size does your algorithm terminate within one day? Which input size can it maximally process in 10 days? Answer these (two!) questions for the following runtimes:

(a) \( f(n) = n \)
(b) \( f(n) = n^2 \)
(c) \( f(n) = 2^n \)
(d) \( f(n) = n^2 + n \)
(e) (Extra, not mandatory) \( f(n) = n \log n \)

Hint: To compute the value of \( f^{-1} \), you can implement the bisection method.

Exercise 8.2 (Big-O)
Consider the Turing machine below. The input alphabet is \( \Sigma = \mathbb{N} = \{1, 2, 3, \ldots\} \). The operator \( |w| \) denotes the length of the string \( w \), the relation \( < \) is the smaller relation on the natural numbers.

\[
M = \text{On input string } w: \\
\text{for } i = 1 \text{ to } |w| \\
\text{for } j = |w| \text{ downto } i + 1 \\
\text{if } w_j < w_{j-1} \\
\text{swap } w_j \text{ and } w_{j-1} \\
\text{endif} \\
\text{endfor} \\
\text{endfor}
\]

Assume that the runtime of a swap and of a comparison of two natural numbers is constant.

(a) What is the smallest exponent \( k \in \mathbb{R} \) so that the runtime of the Turing machine \( M \) is in \( O(|w|^k) \)? Justify your answer.

(b) What does \( M \) compute (i.e. what is written on the tape when \( M \) halts)?

Exercise 8.3 (Big-O)
Characterise the relationship between \( f(n) \) and \( g(n) \) in the following examples using the \( O, \Theta \) or \( \Omega \)-notation.

1. \( f(n) = n^{0.99998} \) \( g(n) = \sqrt{n} \)
2. \( f(n) = 2^\log^2(n) \) \( g(n) = \sum_{k=1}^{n^2} \frac{n}{2^k} \)
3. \( f(n) = n \cdot \log_2 n \) \( g(n) = \sqrt{n} \)
4. \( f(n) = \sqrt{n} \) \( g(n) = 1000n \)
5. (Extra, not mandatory) \( f(n) = \frac{n^{n+1}}{(n+1)!}, g(n) = \sqrt{n!} \)

Hint: Stirling’s approximation could be useful here.