Theoretical Computer Science (Bridging Course)

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Exercise Sheet 0

Exercise 0.1 (Proof by contradiction)

Prove the following statement by *contradiction*

> Let $q \in \mathbb{Q}$ and $x \in \mathbb{R} \setminus \mathbb{Q}$, then $q - x \in \mathbb{R} \setminus \mathbb{Q}$.

That is, the difference of any rational number and any irrational number is irrational.

Solution

We want to derive a contradiction. For this, we assume that $q - x$ is a rational number. According to the definition of a rational number, the following statements hold

$$\begin{aligned} q &= \tfrac{a}{b} \quad \text{for some integer } a, b \text{ such that } b \neq 0 \\ q - x &= \tfrac{c}{d} \quad \text{for some integer } c, d \text{ such that } d \neq 0 \end{aligned}$$

By substitution, we have

$$\begin{aligned} q - x &= \frac{c}{d} \\ \frac{a}{b} - x &= \frac{c}{d} \\ x &= \frac{a}{b} - \frac{c}{d} \\ &= \frac{ad - bc}{bd} \end{aligned}$$

Note that $ad - bc$ is integer as it is obtained as sum and/or product of integers ($a, c \in \mathbb{Z}$, $b, d \in \mathbb{Z} \setminus \{0\}$). Moreover, $bd \neq 0$, since $b \neq 0$ and $d \neq 0$. Therefore, by definition of rational numbers, x is rational. This contradicts our assumption and concludes the proof.

Note: One could show the statement above even more quickly. If p, q are two rational numbers, so is their difference. Hence, if q and $q - x$ were rational, $q - (q - x) = x$ would be rational too, which leads to a contradiction.

Exercise 0.2 (Proofs by induction)

Prove *by induction* that the following statements hold for every $n \in \mathbb{N}^+$ (the set of positive integers).

- $\sum_{i=1}^{n} i^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$
- $1 - x^n = (1 - x)(1 + x + ... + x^{n-1})$.

Please make clear what is the *base case*, the *induction hypothesis* and the *induction step*.

Solution

- First statement.
  - Basis $n=1$: $\sum_{i=1}^{n} i^2 = 1^2 = 1 = \frac{1\cdot(1+1)\cdot(2\cdot 1+1)}{6}$
  - Induction hypothesis: For $n-1$ it holds that $\sum_{i=1}^{n-1} i^2 = \frac{(n-1)\cdot n\cdot(2n-1)}{6}$.
  - Induction step:

$$\begin{aligned}\sum_{i=1}^{n} i^2 &= \left(\sum_{i=1}^{n-1} i^2\right) + n^2 \\ &= \frac{(n-1)n(2n-1)}{6} + n^2 \\ &= \frac{2n^3 - 3n^2 + n + 6n^2}{6} \\ &= \frac{2n^3 + 3n^2 + n}{6} \\ &= \frac{n(n+1)(2n+1)}{6}\end{aligned}$$

- Second statement.
  - Basis $n=1$: trivial.
  - induction hypothesis: For $n-1$, it holds that $1 - x^{n-1} = (1-x)(1 + x + ... + x^{n-2})$.
  - Induction step:

$$\begin{aligned}(1-x)(1+x+...+x^{n-1}) &= (1-x)(1+x+...+x^{n-2}) + (1-x)x^{n-1} \\ &= 1 - x^{n-1} + x^{n-1} - x^n = \\ &= 1 - x^n.\end{aligned}$$

Exercise 0.3 (Sets)
Let $E_1, ..., E_N$ be an arbitrary finite collection of sets. Show that

$$F \cup \left(\bigcap_{n=1}^{N} E_n\right) = \bigcap_{n=1}^{N} (F \cup E_n).$$

Solution
To show that two sets X, Y are equal, a standard approach is to show that both sets are subsets one of the other, that is, $X \subseteq Y$ and $Y \subseteq X$ as well.

($\subseteq$) To prove this bit we need to show that each element of $F \cup \left(\bigcap_{n=1}^{N} E_n\right)$ is also an element of $\bigcap_{n=1}^{N}(F \cup E_n)$. To see this, let $x \in F \cup \left(\bigcap_{n=1}^{N} E_n\right)$, then only two cases are possible:

(i) $x \in F$, then $x \in F \cup E_1, ..., F \cup E_N$ and so it belongs to their intersection.

(ii) $x \notin F$, that is, x must be an element of $\bigcap_{n=1}^{N} E_n$. As a consequence, x lies in every E_i ($i = 1, ..., N$) and thus $x \in F \cup E_1, ..., F \cup E_N$. That is x belongs to the intersection of $F \cup E_i$ ($i = 1, ..., N$).

($\supseteq$) To show the other inclusion we proceed similarly. Let x be an element in $\bigcap_{n=1}^{N}(F \cup E_n)$. Again, only the two scenarios described above must be discussed:

(i) $x \in F$. In such case, it is trivial to see that x belongs to $F \cup \left(\bigcap_{n=1}^{N} E_n \right)$.

(ii) $x \notin F$. Then $x \in E_i$ for every $i = 1, ..., N$. Indeed, if there were a set E_k so that $x \notin E_k$, than $x \notin F \cup E_k$ which contradicts the fact that x do belong to $\bigcap_{n=1}^{N}(F \cup E_n)$. This concludes the argument.

Proof is concluded.