Theoretical Computer Science (Bridging Course)

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Exercise Sheet 10

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Exercise 10.1 (Propositional Logic)

Determine the validity or invalidity of the following argument:

"If Alice is elected class-president, then either Betty is elected vice-president, or Carol is elected treasurer. Betty is elected vice-president. Therefore if Alice is elected class-president, then Carol is not elected treasurer."

Please explain every formal step.

Solution: We use the following symbols for each sentence.

A - Alice is elected class-president
B - Betty is elected vice - president
C - Carol is elected treasurer

The translation for each line of the argument is as follows

$A \to ((B \land \neg C) \lor (\neg B \land C))$	If Alice is elected class-president, then either Betty is elected vice-president, or Carol is elected treasurer.
B	Betty is elected vice-president
$A \to \neg C$	if Alice is elected class-president, then Carol is not elected treasurer.

The sentence corresponding to the argument is

$$\phi := ((A \to ((B \land \neg C) \lor (\neg B \land C))) \land B) \to (A \to \neg C)$$

In order to see if ϕ is valid or not, we can try to find an intepretation I for A, B, C that falsifies ϕ. Looking at the truth table of ϕ we have:

A	B	C	ϕ
T	**T**	**T**	**T**
T	**T**	**F**	**T**
T	**F**	**T**	**T**
T	**F**	**F**	**T**
F	**T**	**T**	**T**
F	**T**	**F**	**T**
F	**F**	**T**	**T**
F	**F**	**F**	**T**

The above statement shows that every interpretation is a model so the argument ϕ is valid. [1]

[1] In this exercise we have used the "exclusive or" (called XOR operator) that is often denoted with $\oplus$: $A \oplus B = (A \land \neg B) \lor (\neg A \land B)$.

Exercise 10.2 (Propositional Logic)

(a) Consider the following logical formula:

$$\phi = (A \leftrightarrow \neg B) \wedge \neg(C \vee B \rightarrow A)$$

Show that $\phi \equiv \neg A \wedge B$ by using the equivalences from the lectures (see slide 17, `08.pdf`) and the equivalences $\psi \wedge \neg\psi \equiv \bot$ and $\psi \vee \bot \equiv \psi \equiv \bot \vee \psi$. Apply in each step only one of the equivalences with the exception that you *may* implicitly use associativity.

Solution:

$$\begin{aligned}
\phi &\equiv (A \leftrightarrow \neg B) \wedge \neg(C \vee B \rightarrow A) && \text{(definition)}\\
&\equiv (A \leftrightarrow \neg B) \wedge \neg(\neg(C \vee B) \vee A) && (\rightarrow \text{ elimination})\\
&\equiv (A \leftrightarrow \neg B) \wedge (\neg\neg(C \vee B) \wedge \neg A) && \text{(De Morgan)}\\
&\equiv (A \leftrightarrow \neg B) \wedge (C \vee B) \wedge \neg A && \text{(double negation)}\\
&\equiv (A \rightarrow \neg B) \wedge (\neg B \rightarrow A) \wedge (C \vee B) \wedge \neg A && (\leftrightarrow \text{ elimination})\\
&\equiv (\neg A \vee \neg B) \wedge (\neg B \rightarrow A) \wedge (C \vee B) \wedge \neg A && (\rightarrow \text{ elimination})\\
&\equiv (\neg A \vee \neg B) \wedge (\neg\neg B \vee A) \wedge (C \vee B) \wedge \neg A && (\rightarrow \text{ elimination})\\
&\equiv (\neg A \vee \neg B) \wedge (B \vee A) \wedge (C \vee B) \wedge \neg A && \text{(double negation)}\\
&\equiv \neg A \wedge (\neg A \vee \neg B) \wedge (B \vee A) \wedge (C \vee B) && \text{(commutativity)}\\
&\equiv \neg A \wedge (B \vee A) \wedge (C \vee B) && \text{(absorption)}\\
&\equiv ((\neg A \wedge B) \vee (\neg A \wedge A)) \wedge (C \vee B) && \text{(distributivity)}\\
&\equiv ((\neg A \wedge B) \vee \bot) \wedge (C \vee B) && (\phi \wedge \neg\phi \equiv \bot)\\
&\equiv \neg A \wedge B \wedge (C \vee B) && (\phi \vee \bot \equiv \phi)\\
&\equiv \neg A \wedge B \wedge (B \vee C) && \text{(commutativity)}\\
&\equiv \neg A \wedge B && \text{(absorption)}
\end{aligned}$$

(b) Consider a vocabulary with only four atomic propositions A, B, C, D. How many models are there for the following formulae? Explain.

i) $(A \wedge B) \vee (B \wedge C)$

ii) $(A \leftrightarrow B) \wedge (B \leftrightarrow C)$

Solution: These can be computed by counting the rows in a truth table that come out true. Remember to count the propositions that are not mentioned; if a sentence mentions only A and B, then we multiply the number of models for $\{A, B\}$ by 2^2 to account for C, D. Hence,

i) Considering that proposition D is not mentioned, there are $3 \cdot 2 = 6$ models that satisfy this formula.

A	B	C	$A \wedge B$	$B \wedge C$	$(A \wedge B) \vee (B \wedge C)$
F	**F**	**F**	**F**	**F**	**F**
F	**F**	**T**	**F**	**F**	**F**
F	**T**	**F**	**F**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**	**F**
T	**F**	**T**	**F**	**F**	**F**
T	**T**	**F**	**T**	**F**	**T**
T	**T**	**T**	**T**	**T**	**T**

ii) Similarly, there are $2 \cdot 2$ models that satisfy this formula.

A	B	C	$(A \leftrightarrow B) \wedge (B \leftrightarrow C)$
F	**F**	**F**	**T**
F	**F**	**T**	**F**
F	**T**	**F**	**F**
F	**T**	**T**	**F**
T	**F**	**F**	**F**
T	**F**	**T**	**F**
T	**T**	**F**	**F**
T	**T**	**T**	**T**

Exercise 10.3 (Propositional Logic)
Show that the following formula is *valid*:

$$(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A).$$

The implication $\neg B \rightarrow \neg A$ is sometimes called *contrapositive* or *counternominal* implication of $A \rightarrow B$.

Solution:
To show the validity of the above formula, we can apply the usual equivalences. We get the following identities:

$$\begin{aligned}
&(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A) \equiv \\
\equiv\ &(\neg A \vee B) \leftrightarrow (\neg\neg B \vee \neg A) \equiv \\
\equiv\ &(\neg A \vee B) \leftrightarrow (B \vee \neg A) \equiv \\
\equiv\ &(\neg A \vee B) \leftrightarrow (\neg A \vee B) \equiv \\
\equiv\ &(\neg(\neg A \vee B) \vee (\neg A \vee B)) \wedge (\neg(\neg A \vee B) \vee (\neg A \vee B)) \equiv \\
\equiv\ &\top \wedge \top \equiv \top
\end{aligned}$$

Equivalently, we could have used a truth table, obtaining

A	B	$(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**T**