Foundations of Artificial Intelligence

8. Satisfiability and Model Construction

Davis-Putnam-Logemann-Loveland Procedure, Phase Transitions, GSAT

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Contents

Motivation

propositional logic - typical algorithmic questions:

- Logical deduction
  - Given: A logical theory (set of propositions)
  - Question: Does a proposition logically follow from this theory?
  - Reduction to unsatisfiability, which is coNP-complete (complementary to NP problems)
- Satisfiability of a formula (SAT)
  - Given: A logical theory
  - Wanted: Model of the theory
  - Example: Configurations that fulfill the constraints given in the theory
  - Can be “easier” because it is enough to find one model

The Satisfiability Problem (SAT)

given:
propositional formula φ in CNF
wanted:

- model of φ
- or proof, that no such model exists

SAT and CSP

SAT can be formulated as a Constraint-Satisfaction-Problem ($\rightarrow$ search):

- CSP-Variables $=$ Symbols of the alphabet
- domain of values $= \{T, F\}$
- constraints given by clauses

The DPLL algorithm

The DPLL algorithm (Davis, Putnam, Logemann, Loveland, 1962) corresponds to backtracking with inference in CPSs:

- recursive Call DPLL (Δ, l) with Δ: set of clauses and l: variable assignment
- result is a satisfying assignment that extends l or 'unsatisfiable' if no such assignment exists.
- first call by DPLL$(\Delta, \emptyset)$

Inference in DPLL:

- simplify: if variable v is assigned a value d, then all clauses containing v are simplified immediately (corresponds to forward checking)
- variables in unit clauses ($=$ clauses with only one variable) are immediately assigned (corresponds to minimum remaining values ordering in CSPs)

The DPLL Procedure

DPLL Function

Given a set of clauses Δ defined over a set of variables Σ, return "satisfiable" if Δ is satisfiable. Otherwise return "unsatisfiable".

1. If $\Delta = \emptyset$ return "satisfiable"
2. If $\square \in \Delta$ return "unsatisfiable"
3. Unit-propagation Rule: If Δ contains a unit-clause C, assign a truth-value to the variable in C that satisfies C, simplify Δ to Δ' and return $\mathrm{DPLL}(\Delta')$.
4. Splitting Rule: Select from Σ a variable v which has not been assigned a truth-value. Assign one truth value t to it, simplify Δ to Δ' and call $\mathrm{DPLL}(\Delta')$
   a. If the call returns "satisfiable", then return "satisfiable".
   b. Otherwise assign *the other* truth-value to v in Δ, simplify to Δ'' and return $\mathrm{DPLL}(\Delta'')$.

Example (1)

$$\Delta = \{\{a, b, \neg c\}, \{\neg a, \neg b\}, \{c\}, \{a, \neg b\}\}$$

1. Unit-propagation rule: $c \mapsto T$
   $\{\{a, b\}, \{\neg a, \neg b\}, \{a, \neg b\}\}$
2. Splitting rule:

2a. $a \mapsto F$
$\{\{b\}, \{\neg b\}\}$

3a. Unit-propagation rule: $b \mapsto T$
$\{\square\}$

2b. $a \mapsto T$
$\{\{\neg b\}\}$

3b. Unit-propagation rule: $b \mapsto F$
$\{\}$

Example (2)

$$\Delta = \{\{a, \neg b, \neg c, \neg d\}, \{b, \neg d\}, \{c, \neg d\}, \{d\}\}$$

1. Unit-propagation rule: $d \mapsto T$
   $\{\{a, \neg b, \neg c\}, \{b\}, \{c\}\}$
2. Unit-propagation rule: $b \mapsto T$
   $\{\{a, \neg c\}, \{c\}\}$
3. Unit-propagation rule: $c \mapsto T$
   $\{\{a\}\}$
4. Unit-propagation rule: $a \mapsto T$
   $\{\}$

Properties of DPLL

- DPLL is complete, correct, and guaranteed to terminate.
- DPLL constructs a model, if one exists.
- In general, DPLL requires exponential time (splitting rule!)
  $\rightarrow$ *Heuristics* are needed to determine which variable should be instantiated next and which value should be used.
- DPLL is polynomial on Horn clauses, i.e., clauses with at most one positive literal $\neg A_1, \vee \ldots \vee \neg A_n \vee B$ (see next slides)
- In all SAT competitions so far, DPLL-based procedures have shown the best performance.

DPLL on Horn Clauses (0)

Horn Clauses constitute an important special case, since they require only polynomial runtime of DPLL.

Definition: A Horn clause is a clause with maximally one positive literal
E.g., $\neg A_1 \vee \ldots \vee \neg A_n \vee B$ or $\neg A_1 \vee \ldots \vee \neg A_n$
($n = 0$ is permitted).

Equivalent representation: $\neg A_1 \vee \ldots \vee \neg A_n \vee B \Leftrightarrow \bigwedge_i A_i \Rightarrow B$
$\rightarrow$ Basis of logic programming (e.g. PROLOG)

DPLL on Horn Clauses (1)

Note:

1. The simplifications in DPLL on Horn clauses always generate Horn clauses
2. If the first sequence of applications of the unit propagation rule in DPLL does not lead to termination, a set of Horn clauses without unit clauses is generated
3. A set of Horn clauses without unit clauses and without the empty clause is satisfiable, since
   - *All clauses have at least one negative literal (since all non-unit clauses have at least two literals, where at most one can be positive (Def. Horn))*
   - *Assigning $false$ to all variables satisfies formula*

DPLL on Horn Clauses (2)

4. It follows from 3.:
   a. every time the splitting rule is applied, the current formula is satisfiable
   b. every time, when the wrong decision (= assignment in the splitting rule) is made, this will be immediately detected (e.g. only through unit propagation steps and the derivation of the empty clause).
4. Therefore, the search trees for n variables can only contain a maximum of n nodes, in which the splitting rule is applied (and the tree branches).
4. Therefore, the size of the search tree is only polynomial in n and therefore the running time is also polynomial.

How Good is DPLL in the Average Case?

- We know that SAT is NP-complete, i.e., in the worst case, it takes exponential time.
- This is clearly also true for the DPLL-procedure.
  $\rightarrow$ Couldn't we do better in the average case?
- For CNF-formulae in which the probability for a positive appearance, negative appearance and non-appearance in a clause is $1/3$, DPLL needs on average quadratic time (Goldberg 79)!
  $\rightarrow$ The probability that these formulae are satisfiable is, however, very high.

Phase Transitions . . .

Conversely, we can, of course, try to identify hard to solve problem instances.

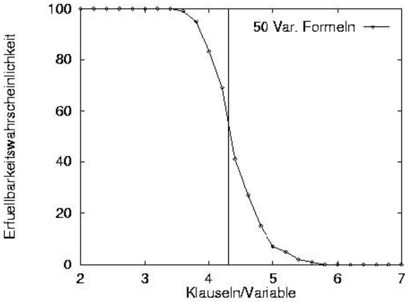
Cheeseman et al. (IJCAI-91) came up with the following plausible conjecture:

> All NP-complete problems have at least *one order* parameter and the hard to solve problems are around a critical value of this order parameter. This critical value (a phase transition) separates one region from another, such as over-constrained and under-constrained regions of the problem space.

Confirmation for graph coloring and Hamilton path . . . later also for other NP-complete problems.

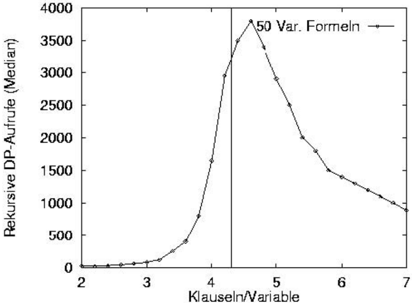
Constant clause length model (Mitchell et al., AAAI-92):
Clause length k is given. Choose variables for every clause k and use the complement with probability 0.5 for each variable.

Phase transition for 3-SAT with a clause/variable ratio of approx. 4.3:



Empirical Difficulty

The Davis-Putnam (DPLL) Procedure shows extreme runtime peaks at the phase transition:



Note: Hard instances can exist even in the regions of the more easily satisfiable/unsatisfiable instances!

Notes on the Phase Transition

- When the probability of a solution is close to 1 (under-constrained), there are many solutions, and the first search path of a backtracking search is usually successful.
- If the probability of a solution is close to 0 (over-constrained), this fact can usually be determined early in the search.
- In the phase transition stage, there are many near successes ("close, but no cigar")

  $\rightarrow$ (limited) possibility of predicting the difficulty of finding a solution based on the parameters

  $\rightarrow$ (search intensive) benchmark problems are located in the phase region (but they have a special structure)

Local Search Methods for Solving Logical Problems

In many cases, we are interested in finding a satisfying assignment of variables (example CSP), and we can sacrifice completeness if we can "solve" much large instances this way.

Standard process for optimization problems: Local Search

- Based on a (random) configuration
- Through local modifications, we hope to produce better configurations

  $\rightarrow$ Main problem: local maxima

Dealing with Local Maxima

As a measure of the value of a configuration in a logical problem, we could use the number of satisfied constraints/clauses.

But local search seems inappropriate, considering we want to find a global maximum (all constraints/clauses satisfied).

By restarting and/or injecting noise, we can often escape local maxima.

Actually: Local search performs very well for finding satisfying assignments of CNF formulae (even without injecting noise).

GSAT

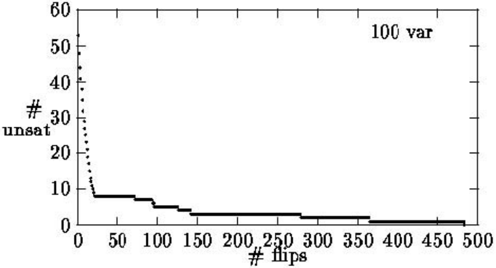
Procedure GSAT

INPUT: a set of clauses α, $\textsc{Max-Flips}$, and $\textsc{Max-Tries}$
OUTPUT: a satisfying truth assignment of α, if found

```
begin
  for i := 1 to MAX-TRIES
    T := a randomly-generated truth assignment
    for j := 1 to MAX-FLIPS
      if T satisfies α then return T
      v := a propositional variable such that a change in its
              truth assignment gives the largest increase in
              the number of clauses of α that are satisfied by T
      T := T with the truth assignment of v reversed
    end for
  end for
  return "no satisfying assignment found"
end
```

The Search Behavior of GSAT

- In contrast to normal local search methods, we must also allow sideways movements!
- Most time is spent searching on plateaus.



State of the Art

- SAT competitions since beginning of the 90s
- Current SAT competitions (`http://www.satcompetition.org/`):
  In 2010:
  - Largest "industrial" instances: $>$ 1,000,000 literals
- Complete solvers are as good as randomized ones on handcrafted and industrial problem

Concluding Remarks

- DPLL-based SAT solvers prevail:
  - Very efficient implementation techniques
  - Good branching heuristics
  - Clause learning
- Incomplete randomized SAT-solvers
  - are good (in particular on random instances)
  - but there is no dramatic increase in size of what they can solve
  - parameters are difficult to adjust