Introduction to Mobile Robotics

Probabilistic Robotics

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Probabilistic Robotics

Key idea:

Explicit representation of uncertainty
(using the calculus of probability theory)

- Perception = state estimation
- Action = utility optimization

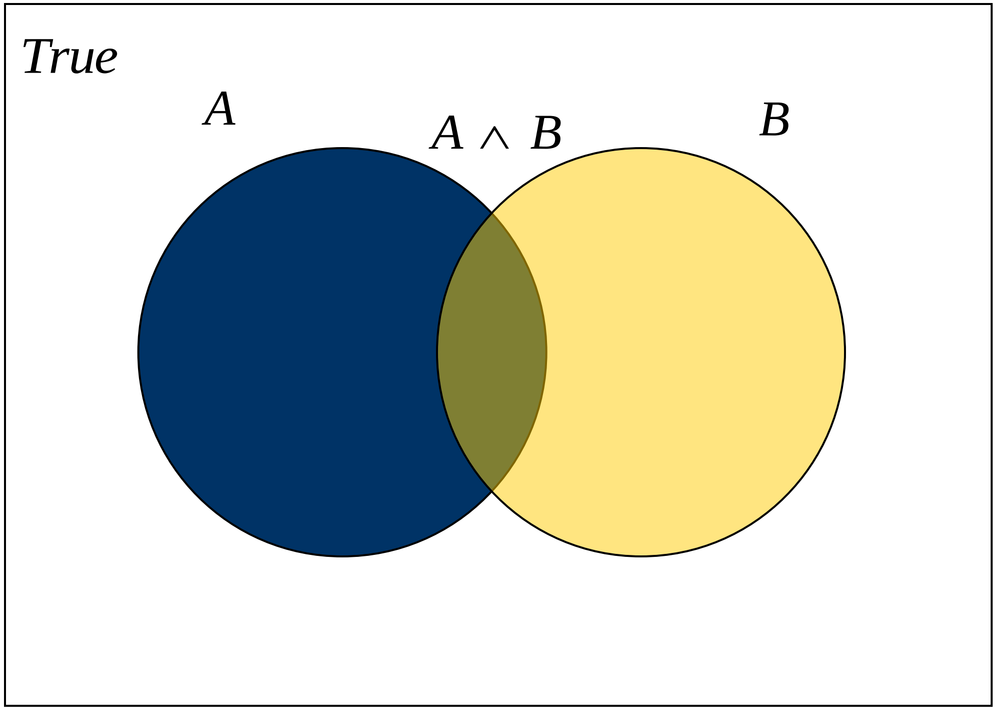
Axioms of Probability Theory

P*(A)* denotes probability that proposition *A* is true.

- $0 \leq \mathrm{P}(A) \leq 1$
- $\mathrm{P}(True) = 1$ $\mathrm{P}(False) = 0$
- $\mathrm{P}(A \vee B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \wedge B)$

A Closer Look at Axiom 3

$$\mathrm{P}(\,A \vee B\,) = \mathrm{P}(\,A\,) + \mathrm{P}(\,B\,) - \mathrm{P}(\,A \wedge B\,)$$



Using the Axioms

$$
\begin{aligned}
\mathrm{P}(A \vee \neg A) &= \mathrm{P}(A) + \mathrm{P}(\neg A) - \mathrm{P}(A \wedge \neg A) \\
\mathrm{P}(\mathit{True}) &= \mathrm{P}(A) + \mathrm{P}(\neg A) - \mathrm{P}(\mathit{False}) \\
1 &= \mathrm{P}(A) + \mathrm{P}(\neg A) - 0 \\
\mathrm{P}(\neg A) &= 1 - \mathrm{P}(A)
\end{aligned}
$$

Discrete Random Variables

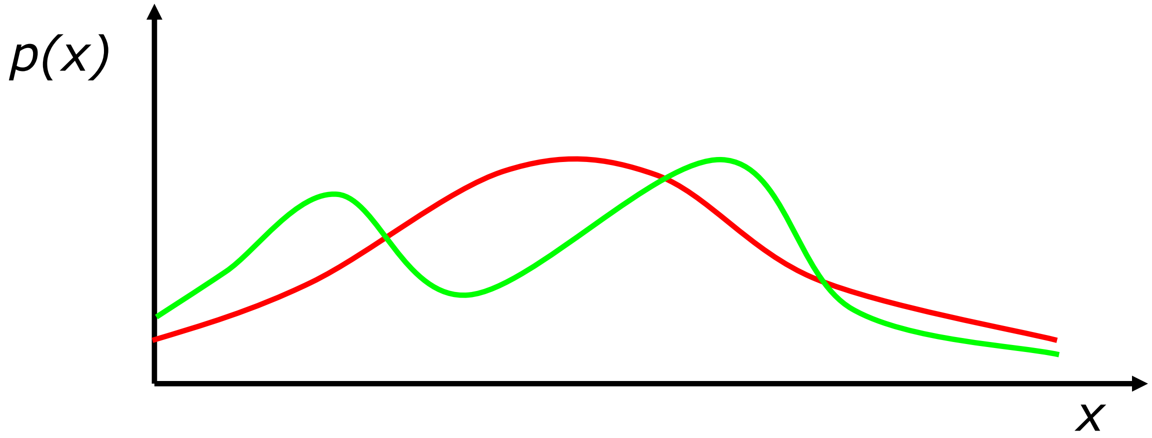
- *X* denotes a random variable
- *X* can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$
- $P(X=x_i)$ or $P(x_i)$ is the probability that the random variable *X* takes on value x_i
- $P(\cdot)$ is called probability mass function
- E.g. $$P(Room) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$$

Continuous Random Variables

- *X* takes on values in the continuum.
- *p(X=x)* or *p(x)* is a probability density function

$$P(x \in [a,b]) = \int_a^b p(x)\,dx$$

- E.g.



"Probability Sums up to One"

Discrete case

$$\sum_{x} P(x) = 1$$

Continuous case

$$\int p(x)dx = 1$$

Joint and Conditional Probability

- $P(X{=}x$ and $Y{=}y) = P(x,y)$

- If X and Y are independent then

$$P(x,y) = P(x)\,P(y)$$

- $P(x \mid y)$ is the probability of x given y

$$P(x \mid y) = P(x,y) \,/\, P(y)$$
$$P(x,y) = P(x \mid y)\,P(y)$$

- If X and Y are independent then

$$P(x \mid y) = P(x)$$

Law of Total Probability

Discrete case

$$P(x) = \sum_y P(x \mid y) P(y)$$

Continuous case

$$p(x) = \int p(x \mid y) p(y) dy$$

Marginalization

Discrete case

$$P(x) = \sum_{y} P(x, y)$$

Continuous case

$$p(x) = \int p(x, y) dy$$

Bayes Formula

$$P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x)$$

$$\Rightarrow$$

$$\boxed{P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}}$$

Normalization

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)} = \eta\, P(y|x)P(x)$$

$$\eta = P(y)^{-1} = \frac{1}{\sum_x P(y|x)P(x)}$$

Algorithm:

$$\forall x : \mathrm{aux}_{x|y} = P(y|x)P(x)$$

$$\eta = \frac{1}{\sum_x \mathrm{aux}_{x|y}}$$

$$\forall x : P(x|y) = \eta\, \mathrm{aux}_{x|y}$$

Bayes Rule with Background Knowledge

$$P(x \mid y, z) = \frac{P(y \mid x, z) P(x \mid z)}{P(y \mid z)}$$

Conditional Independence

$$P(x, y \mid z) = P(x \mid z) P(y \mid z)$$

- Equivalent to $P(x \mid z) = P(x \mid z, y)$

  and $P(y \mid z) = P(y \mid z, x)$

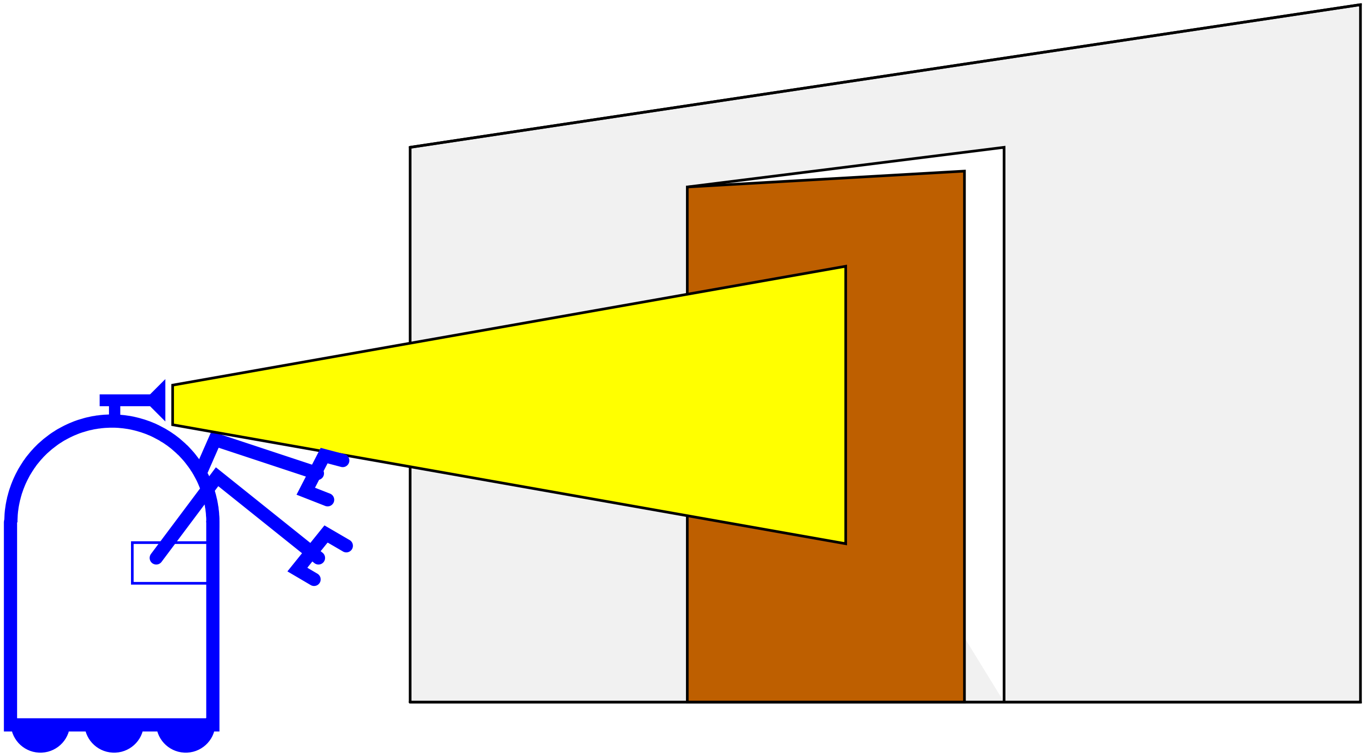
- But this does not necessarily mean

$$P(x, y) = P(x) P(y)$$

(independence/marginal independence)

Simple Example of State Estimation

- Suppose a robot obtains measurement z
- What is *P(open|z)*?



Causal vs. Diagnostic Reasoning

- *P(open|z)* is diagnostic
- *P(z|open)* is causal
- Often causal knowledge is easier to obtain
- Bayes rule allows us to use causal knowledge:

count frequencies!

$$P(open \mid z) = \frac{P(z \mid open)\,P(open)}{P(z)}$$

Example

- $P(z|open) = 0.6 \qquad P(z|\neg open) = 0.3$
- $P(open) = P(\neg open) = 0.5$

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z \mid open)p(open) + P(z \mid \neg open)p(\neg open)}$$

$$P(open \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{0.3}{0.3 + 0.15} = 0.67$$

- *z* raises the probability that the door is open

Combining Evidence

- Suppose our robot obtains another observation z_2

- How can we integrate this new information?

- More generally, how can we estimate $P(x \mid z_1, \ldots, z_n)$?

Recursive Bayesian Updating

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x, z_1, \ldots, z_{n-1}) P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}$$

Markov assumption:

z_n is independent of $z_1, \ldots, z_{n-1}$ if we know x

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x) P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}$$

$$= \eta \, P(z_n \mid x) P(x \mid z_1, \ldots, z_{n-1})$$

$$= \eta_{1 \ldots n} \left[\prod_{i=1 \ldots n} P(z_i \mid x) \right] P(x)$$

Example: Second Measurement

- $P(z_2|open) = 0.25$ $P(z_2|\neg open) = 0.3$
- $P(open|z_1)=2/3$

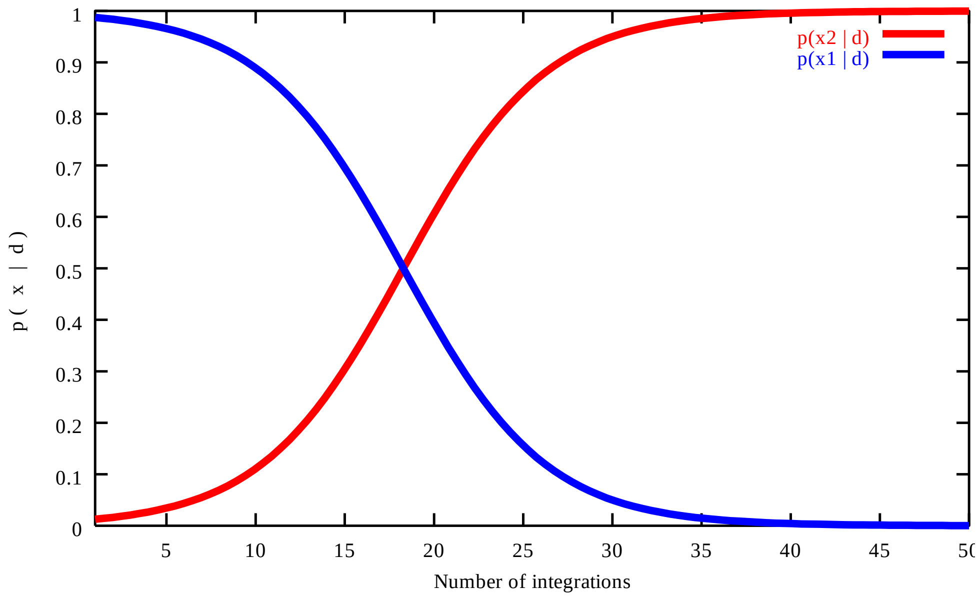
$$P(open \mid z_2, z_1) = \frac{P(z_2 \mid open)P(open \mid z_1)}{P(z_2 \mid open)P(open \mid z_1) + P(z_2 \mid \neg open)P(\neg open \mid z_1)}$$

$$= \frac{\frac{1}{4} \cdot \frac{2}{3}}{\frac{1}{4} \cdot \frac{2}{3} + \frac{3}{10} \cdot \frac{1}{3}} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{10}} = \frac{\frac{1}{6}}{\frac{4}{15}} = \frac{5}{8} = 0.625$$

- z_2 lowers the probability that the door is open

A Typical Pitfall

- Two possible locations x_1 and x_2
- $P(x_1)=0.99$
- $P(z|x_2)=0.09$ $P(z|x_1)=0.07$



Actions

- Often the world is **dynamic** since
  - **actions carried out by the robot**,
  - **actions carried out by other agents**,
  - or just the **time** passing by

  change the world

- How can we **incorporate** such **actions**?

Typical Actions

- The robot **turns its wheels** to move
- The robot **uses its manipulator** to grasp an object
- Plants grow over **time**…

- Actions are **never carried out with absolute certainty**
- In contrast to measurements, **actions generally increase the uncertainty**

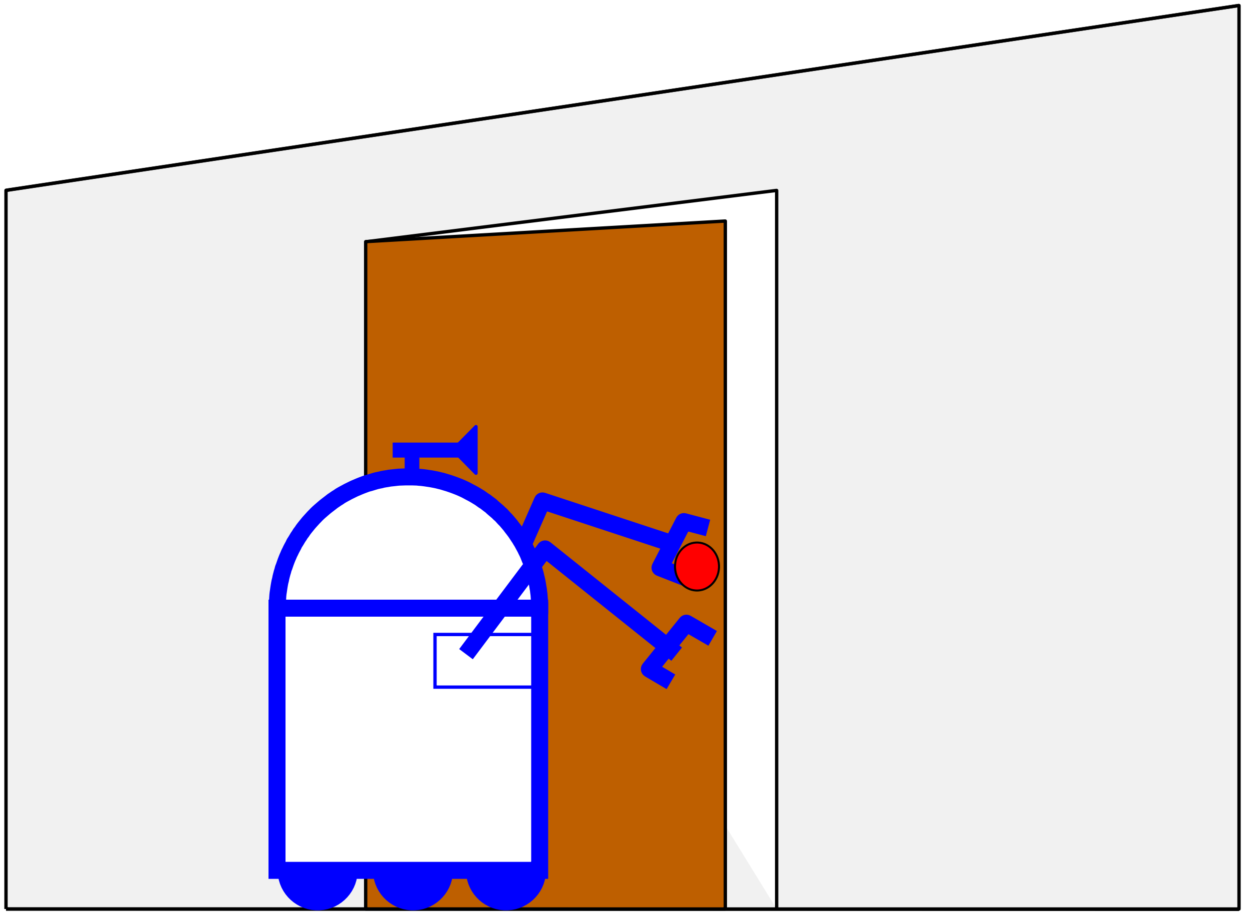
Modeling Actions

- To incorporate the outcome of an action *u* into the current "belief", we use the conditional pdf

$$P(x|u,x')$$

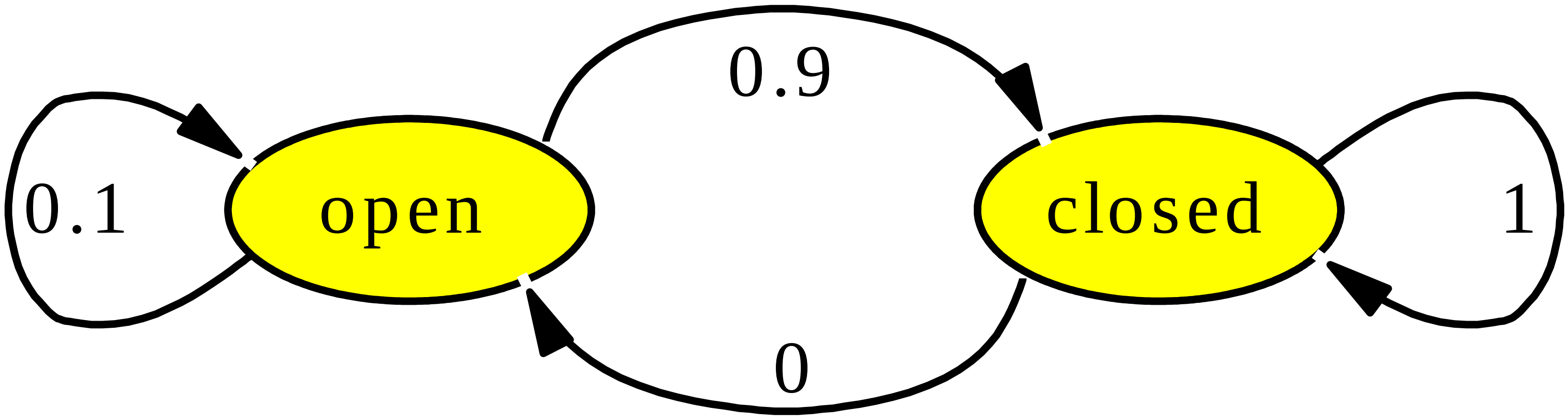
- This term specifies the pdf that **executing *u* changes the state from *x'* to *x***.

Example: Closing the door



State Transitions

$P(x|u,x')$ for u = “close door”:



If the door is open, the action “close door” succeeds in 90% of all cases

Integrating the Outcome of Actions

Continuous case:

$$P(x \mid u) = \int P(x \mid u, x')P(x')dx'$$

Discrete case:

$$P(x \mid u) = \sum P(x \mid u, x')P(x')$$

Example: The Resulting Belief

$$P(closed \mid u) = \sum P(closed \mid u, x')P(x')$$

$$= P(closed \mid u, open)P(open)$$

$$+ P(closed \mid u, closed)P(closed)$$

$$= \frac{9}{10} * \frac{5}{8} + \frac{1}{1} * \frac{3}{8} = \frac{15}{16}$$

$$P(open \mid u) = \sum P(open \mid u, x')P(x')$$

$$= P(open \mid u, open)P(open)$$

$$+ P(open \mid u, closed)P(closed)$$

$$= \frac{1}{10} * \frac{5}{8} + \frac{0}{1} * \frac{3}{8} = \frac{1}{16}$$

$$= 1 - P(closed \mid u)$$

Bayes Filters: Framework

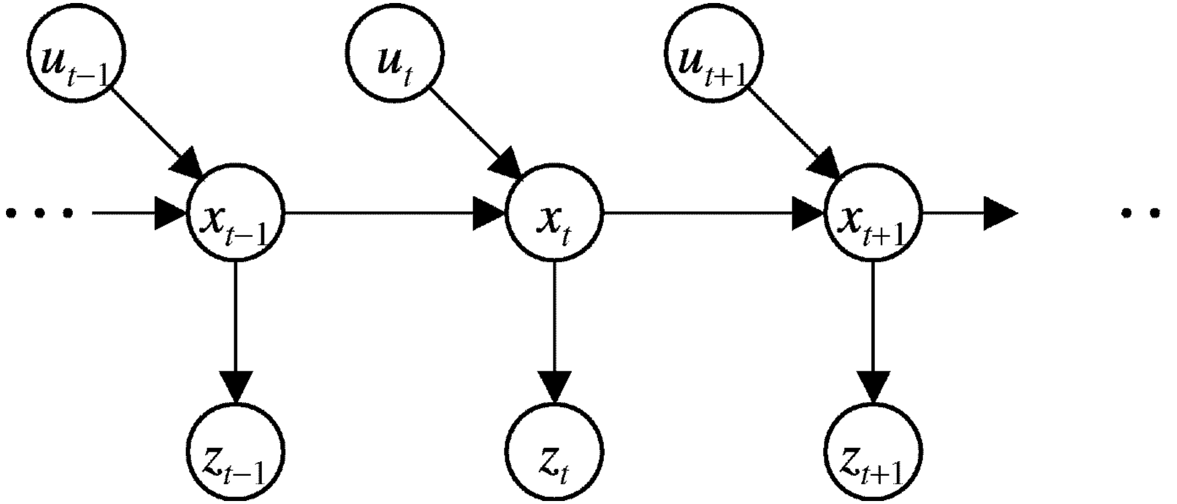
- **Given:**
  - Stream of observations *z* and action data *u:*

$$d_t = \{u_1, z_1 \ldots, u_t, z_t\}$$

  - Sensor model *P(z|x)*
  - Action model *P(x|u,x')*
  - Prior probability of the system state *P(x)*
- **Wanted:**
  - Estimate of the state *X* of a dynamical system
  - The posterior of the state is also called **Belief**:

$$Bel(x_t) = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)$$

Markov Assumption



$$p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$$

$$p(x_t \mid x_{1:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

Bayes Filters

z = observation
u = action
x = state

$$\boxed{Bel(x_t)} = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)$$

Bayes $$= \eta \, P(z_t \mid x_t, u_1, z_1, \ldots, u_t) P(x_t \mid u_1, z_1, \ldots, u_t)$$

Markov $$= \eta \, P(z_t \mid x_t) P(x_t \mid u_1, z_1, \ldots, u_t)$$

Total prob. $$= \eta \, P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots, u_t, x_{t-1}) \, P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \, dx_{t-1}$$

Markov $$= \eta \, P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \, dx_{t-1}$$

Markov $$= \eta \, P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1}) \, dx_{t-1}$$

$$\boxed{= \eta \, P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \, Bel(x_{t-1}) \, dx_{t-1}}$$

$$Bel(x_t) = \eta \, P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \, Bel(x_{t-1}) \, dx_{t-1}$$

Algorithm **Bayes_filter**($Bel(x), d$):

η=0

If d is a perceptual data item z then

    For all x do

        $Bel'(x) = P(z \mid x) \, Bel(x)$

        $\eta = \eta + Bel'(x)$

    For all x do

        $Bel'(x) = \eta^{-1} Bel'(x)$

Else if d is an action data item u then

    For all x do

        $Bel'(x) = \int P(x \mid u, x') \, Bel(x') \, dx'$

Return $Bel'(x)$

Bayes Filters are Familiar!

$$Bel(x_t) = \eta\, P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\, Bel(x_{t-1})\, dx_{t-1}$$

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)

Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.