Introduction to Mobile Robotics

Probabilistic Robotics

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Probabilistic Robotics

Key idea:

Explicit representation of uncertainty
(using the calculus of probability theory)

- Perception = state estimation
- Action = utility optimization
Axioms of Probability Theory

P(A) denotes probability that proposition A is true.

- $0 \leq P(A) \leq 1$
- $P(True) = 1$  \hspace{1cm}  $P(False) = 0$
- $P(A \lor B) = P(A) + P(B) - P(A \land B)$
A Closer Look at Axiom 3

\[ P( A \lor B ) = P( A ) + P( B ) - P( A \land B ) \]
Using the Axioms

\[
P(A \lor \neg A) = P(A) + P(\neg A) - P(A \land \neg A)
\]

\[
P(True) = P(A) + P(\neg A) - P(False)
\]

\[
1 = P(A) + P(\neg A) - 0
\]

\[
P(\neg A) = 1 - P(A)
\]
Discrete Random Variables

- $X$ denotes a random variable
- $X$ can take on a countable number of values in $\{x_1, x_2, ..., x_n\}$
- $P(X=x_i)$ or $P(x_i)$ is the probability that the random variable $X$ takes on value $x_i$
- $P(\cdot)$ is called probability mass function

E.g. $P(Room) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$
Continuous Random Variables

- $X$ takes on values in the continuum.
- $p(X=x)$ or $p(x)$ is a probability density function

$$P(x \in [a, b]) = \int_{a}^{b} p(x) \, dx$$

- E.g.
“Probability Sums up to One”

Discrete case

\[ \sum_x P(x) = 1 \]

Continuous case

\[ \int p(x)\,dx = 1 \]
Joint and Conditional Probability

- \( P(X=x \text{ and } Y=y) = P(x, y) \)

- If \( X \) and \( Y \) are independent then
  \[ P(x, y) = P(x) \cdot P(y) \]

- \( P(x \mid y) \) is the probability of \( x \) given \( y \)
  \[ P(x \mid y) = \frac{P(x, y)}{P(y)} \]
  \[ P(x, y) = P(x \mid y) \cdot P(y) \]

- If \( X \) and \( Y \) are independent then
  \[ P(x \mid y) = P(x) \]
Law of Total Probability

Discrete case

\[ P(x) = \sum_y P(x \mid y)P(y) \]

Continuous case

\[ p(x) = \int p(x \mid y) p(y) dy \]
Marginalization

Discrete case

\[ P(x) = \sum_y P(x, y) \]

Continuous case

\[ p(x) = \int p(x, y) dy \]
Bayes Formula

\[ P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x) \]

\[ \Rightarrow \]

\[ P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}} \]
Normalization

\[ P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)} = \eta P(y \mid x)P(x) \]

\[ \eta = P(y)^{-1} = \frac{1}{\sum_x P(y \mid x)P(x)} \]

Algorithm:

\[ \forall x : \text{aux } x \mid y = P(y \mid x)P(x) \]

\[ \eta = \frac{1}{\sum_x \text{aux } x \mid y} \]

\[ \forall x : P(x \mid y) = \eta \text{aux } x \mid y \]
Bayes Rule
with Background Knowledge

\[ P(x \mid y, z) = \frac{P(y \mid x, z)P(x \mid z)}{P(y \mid z)} \]
Conditional Independence

\[ P(x, y \mid z) = P(x \mid z)P(y \mid z) \]

- Equivalent to \[ P(x \mid z) = P(x \mid z, y) \]

  and \[ P(y \mid z) = P(y \mid z, x) \]

- But this does not necessarily mean \[ P(x, y) = P(x)P(y) \]

  (independence/marginal independence)
Simple Example of State Estimation

- Suppose a robot obtains measurement $z$
- What is $P(open|z)$?
Causal vs. Diagnostic Reasoning

- \( P(open|z) \) is diagnostic
- \( P(z|open) \) is causal
- Often causal knowledge is easier to obtain
- Bayes rule allows us to use causal knowledge:

\[
P(open|z) = \frac{P(z|open) \cdot P(open)}{P(z)}
\]
Example

- \( P(z|\text{open}) = 0.6 \quad \text{and} \quad P(z|\neg\text{open}) = 0.3 \)
- \( P(\text{open}) = P(\neg\text{open}) = 0.5 \)

\[
P(\text{open} | z) = \frac{P(z|\text{open})P(\text{open})}{P(z|\text{open})p(\text{open}) + P(z|\neg\text{open})p(\neg\text{open})}
\]

\[
P(\text{open} | z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{0.3}{0.3 + 0.15} = 0.67
\]

- \( z \) raises the probability that the door is open.
Combining Evidence

- Suppose our robot obtains another observation $z_2$

- How can we integrate this new information?

- More generally, how can we estimate $P(x \mid z_1, \ldots, z_n)$?
Recursive Bayesian Updating

\[
P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x, z_1, \ldots, z_{n-1})P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}
\]

Markov assumption:
\(z_n\) is independent of \(z_1, \ldots, z_{n-1}\) if we know \(x\)

\[
P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x)P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})}
= \eta P(z_n \mid x)P(x \mid z_1, \ldots, z_{n-1})
= \eta \prod_{i=1}^{n} P(z_i \mid x) P(x)
\]
Example: Second Measurement

- \( P(z_2|\text{open}) = 0.25 \)
- \( P(z_2|\neg\text{open}) = 0.3 \)
- \( P(\text{open}|z_1)=2/3 \)

\[
P(\text{open} \mid z_2, z_1) = \frac{P(z_2 \mid \text{open})P(\text{open} \mid z_1)}{P(z_2 \mid \text{open})P(\text{open} \mid z_1) + P(z_2 \mid \neg\text{open})P(\neg\text{open} \mid z_1)}
\]

\[
= \frac{\frac{1}{4} \cdot \frac{2}{3}}{\frac{1}{4} \cdot \frac{2}{3} + \frac{1}{6} \cdot \frac{1}{10}} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{15}} = \frac{\frac{5}{8}}{\frac{4}{15}} = 0.625
\]

- \( z_2 \) lowers the probability that the door is open
A Typical Pitfall

- Two possible locations $x_1$ and $x_2$
- $P(x_1) = 0.99$
- $P(z|x_2) = 0.09$ $P(z|x_1) = 0.07$
**Actions**

- Often the world is **dynamic** since
  - actions carried out by the robot,
  - actions carried out by other agents,
  - or just the **time** passing by

- How can we **incorporate** such **actions**?
Typical Actions

- The robot **turns its wheels** to move
- The robot **uses its manipulator** to grasp an object
- Plants grow over **time**...

- Actions are **never carried out with absolute certainty**
- In contrast to measurements, **actions generally increase the uncertainty**
To incorporate the outcome of an action $u$ into the current “belief”, we use the conditional pdf

$$P(x|u, x')$$

This term specifies the pdf that executing $u$ changes the state from $x'$ to $x$. 
Example: Closing the door
State Transitions

\[ P(x|u,x') \] for \( u = \text{“close door”} \):

If the door is open, the action “close door” succeeds in 90% of all cases.
Integrating the Outcome of Actions

Continuous case:

\[ P(x \mid u) = \int P(x \mid u, x') P(x') \, dx' \]

Discrete case:

\[ P(x \mid u) = \sum P(x \mid u, x') P(x') \]
Example: The Resulting Belief

\[
P(\text{closed} \mid u) = \sum P(\text{closed} \mid u, x') P(x')
\]
\[
= P(\text{closed} \mid u, \text{open}) P(\text{open})
\]
\[
+ P(\text{closed} \mid u, \text{closed}) P(\text{closed})
\]
\[
= \frac{9}{10} \times \frac{5}{8} + \frac{1}{1} \times \frac{3}{8} = \frac{15}{16}
\]

\[
P(\text{open} \mid u) = \sum P(\text{open} \mid u, x') P(x')
\]
\[
= P(\text{open} \mid u, \text{open}) P(\text{open})
\]
\[
+ P(\text{open} \mid u, \text{closed}) P(\text{closed})
\]
\[
= \frac{1}{10} \times \frac{5}{1} + \frac{0}{1} \times \frac{3}{8} = \frac{1}{16}
\]
\[
= 1 - P(\text{closed} \mid u)
\]
Bayes Filters: Framework

- **Given:**
  - Stream of observations $z$ and action data $u$:
    \[ d_t = \{u_1, z_1 \ldots, u_t, z_t\} \]
  - Sensor model $P(z|x)$
  - Action model $P(x|u,x')$
  - Prior probability of the system state $P(x)$

- **Wanted:**
  - Estimate of the state $X$ of a dynamical system
  - The posterior of the state is also called **Belief**:
    \[ Bel(x_t) = P(x_t | u_1, z_1 \ldots, u_t, z_t) \]
Markov Assumption

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors
Bayes Filters

\[ \text{Bel} \left( x_t \right) = P \left( x_t \mid u_1, z_1 \ldots, u_t, z_t \right) \]

Bayes
\[ = \eta P \left( z_t \mid x_t, u_1, z_1, \ldots, u_t \right) P \left( x_t \mid u_1, z_1, \ldots, u_t \right) \]

Markov
\[ = \eta P \left( z_t \mid x_t \right) P \left( x_t \mid u_1, z_1, \ldots, u_t \right) \]

Total prob.
\[ = \eta P \left( z_t \mid x_t \right) \int P \left( x_t \mid u_1, z_1, \ldots, u_t, x_{t-1} \right) \]
\[ \quad \times P \left( x_{t-1} \mid u_1, z_1, \ldots, u_t \right) dx_{t-1} \]

Markov
\[ = \eta P \left( z_t \mid x_t \right) \int P \left( x_t \mid u_t, x_{t-1} \right) P \left( x_{t-1} \mid u_1, z_1, \ldots, u_t \right) dx_{t-1} \]

Markov
\[ = \eta P \left( z_t \mid x_t \right) \int P \left( x_t \mid u_t, x_{t-1} \right) P \left( x_{t-1} \mid u_1, z_1, \ldots, z_{t-1} \right) dx_{t-1} \]

\[ = \eta P \left( z_t \mid x_t \right) \int P \left( x_t \mid u_t, x_{t-1} \right) \text{Bel} \left( x_{t-1} \right) dx_{t-1} \]
Bayes Filter Algorithm

1. Algorithm \textbf{Bayes\_filter}(Bel(x), d):
2. \( \eta = 0 \)
3. If \( d \) is a perceptual data item \( z \) then
4. For all \( x \) do
5. \( Bel'(x) = P(z|x)Bel(x) \)
6. \( \eta = \eta + Bel'(x) \)
7. For all \( x \) do
8. \( Bel'(x) = \eta^{-1}Bel'(x) \)
9. Else if \( d \) is an action data item \( u \) then
10. For all \( x \) do
11. \( Bel'(x) = \int P(x|u,x')Bel(x')dx' \)
12. Return \( Bel'(x) \)

\[
Bel(x_t) = \eta P(z_t | x_t) \int P(x_t | u_t, x_{t-1})Bel(x_{t-1})dx_{t-1}
\]
Bayes Filters are Familiar!

\[ Bel(x_t) = \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1} \]

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)
Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.