Foundations of Artificial Intelligence 5. Constraint Satisfaction Problems CSPs as Search Problems, Solving CSPs, Problem Structure

Joschka Boedecker and Wolfram Burgard and Bernhard Nebel



Albert-Ludwigs-Universität Freiburg

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#### 1 What are CSPs?

- 2 Backtracking Search for CSPs
- 3 CSP Heuristics
- 4 Constraint Propagation
- 5 Problem Structure

- A Constraint Satisfaction Problems (CSP) is given by
  - a set of variables  $\{x_1, x_2, \ldots, x_n\}$ ,
  - an associated set of value domains  $\{dom_1, dom_2, \dots, dom_n\}$ , and
  - a set of constraints. i.e., relations, over the variables.
  - An assignment of values to variables that satisfies all constraints is a solution of such a CSP.
- In CSPs viewed as search problems, states are explicitly represented as variable assignments. CSP search algorithms take advantage of this structure.
- The main idea is to exploit the constraints to eliminate large portions of search space.
- Formal representation language with associated general inference algorithms

#### Example: Map-Coloring



- Variables: WA, NT, SA, Q, NSW, V, T
- Values: {red, green, blue}
- Constraints: adjacent regions must have different colors, e.g.,  $NSW \neq V$

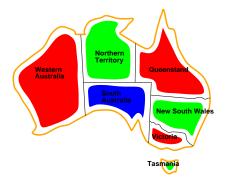
#### Australian Capital Territory (ACT) and Canberra (inside NSW)



View of the Australian National University and Telstra Tower

(University of Freiburg)

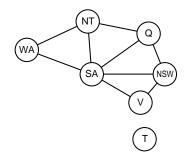
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#### • Solution assignment:

- { WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green }
- Perhaps in addition ACT = blue

#### Constraint Graph



- a constraint graph can be used to visualize binary constraints
- for higher order constraints, hyper-graph representations might be used
- Nodes = variables, arcs = constraints

#### Note: Our problem is three-colorability for a planar graph

- Binary, ternary, or even higher arity (e.g., ALL\_DIFFERENT)
- Finite domains (d values)  $\rightarrow d^n$  possible variable assignments
- Infinite domains (reals, integers)
  - linear constraints (each variable occurs only in linear form): solvable (in P if real)
  - nonlinear constraints: unsolvable

- Timetabling (classes, rooms, times)
- Configuration (hardware, cars, ...)
- Spreadsheets
- Scheduling
- Floor planning
- Frequency assignments
- Sudoku
- . . .

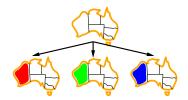
- Assign values to variables step by step (order does not matter)
- Consider only one variable per search node!
- DFS with single-variable assignments is called backtracking search
- Can solve  $n\text{-}{\rm queens}$  for  $n\approx 25$

```
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
  return BACKTRACK({ }, csp)
function BACKTRACK(assignment, csp) returns a solution, or failure
  if assignment is complete then return assignment
  var \leftarrow SELECT-UNASSIGNED-VARIABLE(csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment then
         add \{var = value\} to assignment
         inferences \leftarrow \text{INFERENCE}(csp, var, value)
         if inferences \neq failure then
            add inferences to assignment
            result \leftarrow BACKTRACK(assignment, csp)
            if result \neq failure then
              return result
      remove \{var = value\} and inferences from assignment
  return failure
```

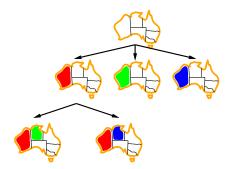




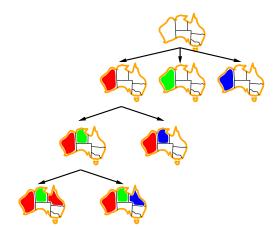












#### Improving Efficiency: CSP Heuristics & Pruning Techniques

- Variable ordering: Which one to assign first?
- Value ordering: Which value to try first?
- Try to detect failures early on
- Try to exploit problem structure
- $\rightarrow$  Note: all this is not problem-specific!

#### Variable Ordering: Most constrained first

- Most constrained variable:
  - choose the variable with the fewest remaining legal values
  - $\rightarrow\,$  reduces branching factor!



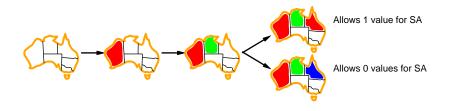
# Variable Ordering: Most Constraining Variable First

- Break ties among variables with the same number of remaining legal values:
  - choose variable with the most constraints on remaining unassigned variables
  - $\rightarrow\,$  reduces branching factor in the next steps



# Value Ordering: Least Constraining Value First

- Given a variable,
  - choose first a value that rules out the fewest values in the remaining unassigned variables
  - $\rightarrow$  We want to find an assignment that satisfies the constraints (of course, this does not help if the given problem is unsatisfiable.)

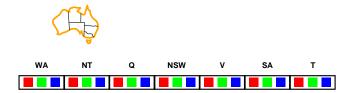


# Rule out Failures early on: Forward Checking

- Whenever a value is assigned to a variable, values that are now illegal for other variables are removed
- Implements what the ordering heuristics implicitly compute
- *WA* = *red*, then *NT* cannot become *red*
- If all values are removed for one variable, we can stop!

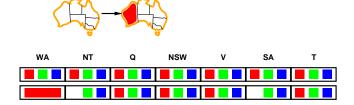
# Forward Checking (1)

- Keep track of remaining values
- Stop if all have been removed



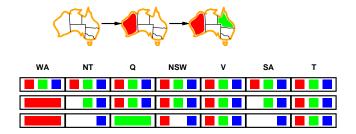
# Forward Checking (2)

- Keep track of remaining values
- Stop if all have been removed



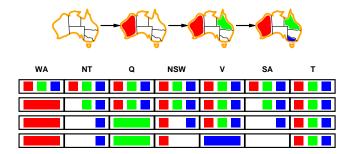
# Forward Checking (3)

- Keep track of remaining values
- Stop if all have been removed



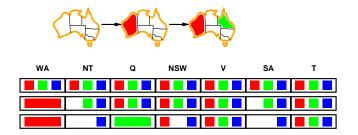
# Forward Checking (4)

- Keep track of remaining values
- Stop if all have been removed



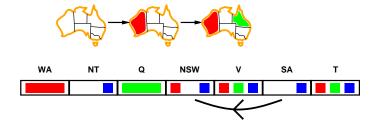
# Forward Checking: Sometimes it Misses Something

- Forward Checking propagates information from assigned to unassigned variables
- However, there is no propagation between unassigned variables



- A directed arc  $X \to Y$  is "consistent" iff
  - for every value x of X, there exists a value y of Y, such that (x,y) satisfies the constraint between X and Y
- ${\ensuremath{\, \circ }}$  Remove values from the domain of X to enforce arc-consistency
- Arc consistency detects failures earlier
- Can be used as preprocessing technique or as a propagation step during backtracking

# Arc Consistency Example



#### AC-3 Algorithm

```
function AC-3(csp) returns false if an inconsistency is found and true otherwise

inputs: csp, a binary CSP with components (X, D, C)

local variables: queue, a queue of arcs, initially all the arcs in csp

while queue is not empty do

(X_i, X_j) \leftarrow \text{REMOVE-FIRST}(queue)

if REVISE(csp, X_i, X_j) then

if size of D_i = 0 then return false

for each X_k in X_i.NEIGHBORS - \{X_j\} do

add (X_k, X_j) to queue
```

```
return true
```

```
function REVISE( csp, X_i, X_j) returns true iff we revise the domain of X_i

revised \leftarrow false

for each x in D_i do

if no value y in D_j allows (x,y) to satisfy the constraint between X_i and X_j then

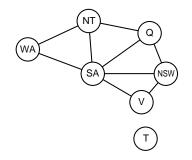
delete x from D_i

revised \leftarrow true

return revised
```

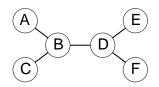
- AC-3 runs in  $O(d^3n^2)$  time, with n being the number of nodes and d being the maximal number of elements in a domain
- Of course, AC-3 does not detect all inconsistencies (which is an NP-hard problem)

#### Problem Structure (1)



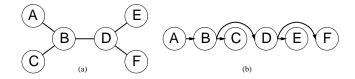
- CSP has two independent components
- Identifiable as connected components of constraint graph
- Can reduce the search space dramatically

#### Problem Structure (2): Tree-structured CSPs



- If the CSP graph is a tree, then it can be solved in  $O(nd^2)$  (general CSPs need in the worst case  $O(d^n)$ ).
- *Idea:* Pick root, order nodes, apply arc consistency from leaves to root, and assign values starting at root.

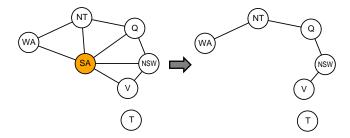
# Problem Structure (2): Tree-structured CSPs



- Pick any variable as root; choose an ordering such that each variable appears after its parent in the tree.
- Apply arc-consistency to  $(x_i, x_k)$  when  $x_i$  is the parent of  $x_k$  for all k = n down to 2 (any tree with n nodes has n 1 arcs and per arc  $d^2$  comparisons are needed, which results in a complexity of  $O(n d^2)$ ).
- Now we can start at  $x_1$  assigning values from the remaining domains without creating any conflict in one sweep through the tree!
- This algorithm is linear in n.

# Problem Structure (3): Almost Tree-structured

Idea: Reduce the graph structure to a tree by fixing values in a reasonably chosen subset



Instantiate a variable and prune values in neighboring variables is called Conditioning

# Problem Structure (4): Almost Tree-structured

Algorithm Cutset Conditioning:

- Choose a subset S of the CSPs variables such that the constraint graph becomes a tree after removal of S. The set S is called a cycle cutset.
- For each possible assignment of variables in S that satisfies all constraints on S
  - remove from the domains of the remaining variables any values that are inconsistent with the assignments for S, and
  - If the remaining CSP has a solution, return it together with the assignment for S



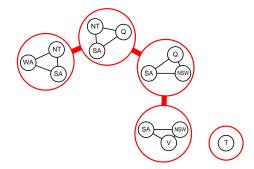
Note: Finding the smallest cycle cutset is NP hard, but several efficient approximation algorithms are known.

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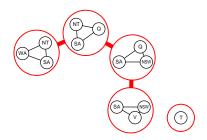
# Another Method: Tree Decomposition (1)

- Decompose the problem into a set of connected sub-problems, where two sub-problems are connected when they share a constraint
- Solve the sub-problems independently and then combine the solutions



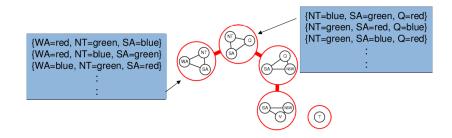
# Another Method: Tree Decomposition (2)

- A tree decomposition must satisfy the following conditions:
  - Every variable of the original problem appears in at least one sub-problem
  - Every constraint appears in at least one sub-problem
  - If a variable appears in two sub-problems, it must appear in all sub-problems on the path between the two sub-problems
  - The connections form a tree



# Another Method: Tree Decomposition (3)

- Consider sub-problems as new mega-variables, which have values defined by the solutions to the sub-problems
- Use technique for tree-structured CSP to find an overall solution (constraint is to have identical values for the same variable)



- The aim is to make the subproblems as small as possible. The tree width w of a tree decomposition is the size of largest sub-problem minus 1
- Tree width of a graph is minimal tree width over all possible tree decompositions
- If a graph has tree width w and we know a tree decomposition with that width, we can solve the problem in  $O(nd^{w+1})$
- Unfortunately, finding a tree decomposition with minimal tree width is NP-hard. However, there are heuristic methods that work well in practice.

#### Summary & Outlook

- CSPs are a special kind of search problem:
  - states are value assignments
  - goal test is defined by constraints
- Backtracking = DFS with one variable assigned per node. Other intelligent backtracking techniques possible
- Variable/value ordering heuristics can help dramatically
- Constraint propagation prunes the search space
- Path-consistency is a constraint propagation technique for triples of variables
- Tree structure of CSP graph simplifies problem significantly
- Cutset conditioning and tree decomposition are two ways to transform part of the problem into a tree
- CSPs can also be solved using local search