Propositional Logic — typical algorithmic questions:

- **Logical deduction**
  - **Given:** A logical theory (set of propositions)
  - **Question:** Does a proposition logically follow from this theory?
  - Reduction to unsatisfiability, which is coNP-complete (complementary to NP problems)

- **Satisfiability of a formula (SAT)**
  - **Given:** A logical theory
  - **Wanted:** Model of the theory
  - **Example:** Configurations that fulfill the constraints given in the theory
  - Can be “easier” because it is enough to find one model
The Satisfiability Problem (SAT)

Given:

Propositional formula $\varphi$ in CNF

Wanted:

- Model of $\varphi$.
- or proof, that no such model exists.
SAT can be formulated as a Constraint-Satisfaction-Problem (→ search):
SAT can be formulated as a Constraint-Satisfaction-Problem (→ search):

- CSP-Variables = Symbols of the alphabet
- Domain of values = \{T, F\}
- Constraints given by clauses
The DPLL algorithm (Davis, Putnam, Logemann, Loveland, 1962) corresponds to backtracking with inference in CPSs:

- recursive Call DPLL ($\Delta, l$) with $\Delta$: set of clauses and $l$: variable assignment
- result is a satisfying assignment that extends $l$ or “unsatisfiable” if no such assignment exists.
- first call by DPLL($\Delta, \emptyset$)

Inference in DPLL:

- simplify: if variable $v$ is assigned a value $d$, then all clauses containing $v$ are simplified immediately (corresponds to forward checking)
- variables in unit clauses (= clauses with only one variable) are immediately assigned (corresponds to minimum remaining values ordering in CSPs)
DPLL Function

Given a set of clauses $\Delta$ defined over a set of variables $\Sigma$, return “satisfiable” if $\Delta$ is satisfiable. Otherwise return “unsatisfiable”.

1. If $\Delta = \emptyset$ return “satisfiable”
2. If $\Box \in \Delta$ return “unsatisfiable”
3. **Unit-propagation Rule**: If $\Delta$ contains a unit-clause $C$, assign a truth-value to the variable in $C$ that satisfies $C$, simplify $\Delta$ to $\Delta'$ and return $\text{DPLL}(\Delta')$.
4. **Splitting Rule**: Select from $\Sigma$ a variable $v$ which has not been assigned a truth-value. Assign one truth value $t$ to it, simplify $\Delta$ to $\Delta'$ and call $\text{DPLL}(\Delta')$
   a. If the call returns “satisfiable”, then return “satisfiable”.
   b. Otherwise assign the other truth-value to $v$ in $\Delta$, simplify to $\Delta''$ and return $\text{DPLL}(\Delta'')$.  

(University of Freiburg)
Example (1)

\[ \Delta = \{\{a, b, \neg c\}, \{\neg a, \neg b\}, \{c\}, \{a, \neg b\}\} \]
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2. Splitting rule:
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2. Splitting rule:

2a. \( a \mapsto F \)
   \( \{\{b\}, \{\neg b\}\} \)
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3a. Unit-propagation rule: \( b \mapsto T \)
   \[ \{\Box\} \]
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   2b. \( a \mapsto T \)
   \[ \{\{\neg b\}\} \]

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3a. Unit-propagation rule: \( b \mapsto T \)
   \[ \{\square\} \]

2b. \( a \mapsto T \)
   \[ \{\{\neg b\}\} \]

3b. Unit-propagation rule: \( b \mapsto F \)
   \[ \{\} \]
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   2a. \( a \mapsto F \)
       \[ \{\{b\}, \{\neg b\}\} \]

   3a. Unit-propagation rule: \( b \mapsto T \)
       \[ \{\square\} \]

   2b. \( a \mapsto T \)
       \[ \{\{\neg b\}\} \]

   3b. Unit-propagation rule: \( b \mapsto F \)
       \[ \{\} \]
\[ \Delta = \{ \{ a, \neg b, \neg c, \neg d \}, \{ b, \neg d \}, \{ c, \neg d \}, \{ d \} \} \]
Example (2)

\[ \Delta = \{ \{ a, \neg b, \neg c, \neg d \}, \{ b, \neg d \}, \{ c, \neg d \}, \{ d \} \} \]

1. **Unit-propagation rule:** \( d \mapsto T \)
Example (2)

\[ \Delta = \{\{a, \neg b, \neg c, \neg d\}, \{b, \neg d\}, \{c, \neg d\}, \{d\}\} \]

1. Unit-propagation rule: \( d \mapsto T \)
   \[ \{\{a, \neg b, \neg c\}, \{b\}, \{c\}\} \]
Example (2)

\[
\Delta = \{\{a, \neg b, \neg c, \neg d\}, \{b, \neg d\}, \{c, \neg d\}, \{d\}\}
\]

1. Unit-propagation rule: \(d \mapsto T\)
   \(\{\{a, \neg b, \neg c\}, \{b\}, \{c\}\}\)

2. Unit-propagation rule: \(b \mapsto T\)
   \(\{\{a, \neg c\}, \{c\}\}\)
Example (2)

\[ \Delta = \{ \{a, \neg b, \neg c, \neg d\}, \{b, \neg d\}, \{c, \neg d\}, \{d\} \} \]

1. Unit-propagation rule: \( d \mapsto T \)
   \( \{\{a, \neg b, \neg c\}, \{b\}, \{c\}\} \)

2. Unit-propagation rule: \( b \mapsto T \)
   \( \{\{a, \neg c\}, \{c\}\} \)

3. Unit-propagation rule: \( c \mapsto T \)
   \( \{\{a\}\} \)
Example (2)

\[ \Delta = \{ \{a, \neg b, \neg c, \neg d\}, \{b, \neg d\}, \{c, \neg d\}, \{d\} \} \]

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   \[ \{ \{a, \neg b, \neg c\}, \{b\}, \{c\} \} \]

2. Unit-propagation rule: \( b \mapsto T \)
   \[ \{ \{a, \neg c\}, \{c\} \} \]

3. Unit-propagation rule: \( c \mapsto T \)
   \[ \{ \{a\} \} \]

4. Unit-propagation rule: \( a \mapsto T \)
   \[ \{\} \]
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3. Unit-propagation rule: \( c \mapsto T \)
   \[ \{ \{a\} \} \]

4. Unit-propagation rule: \( a \mapsto T \)
   \[ \{\} \]
Properties of DPLL

- DPLL is complete, correct, and guaranteed to terminate.
- DPLL constructs a model, if one exists.
- In general, DPLL requires exponential time (splitting rule!) → Heuristics are needed to determine which variable should be instantiated next and which value should be used.
- DPLL is polynomial on Horn clauses, i.e., clauses with at most one positive literal \( \neg A_1, \lor \ldots \lor \neg A_n \lor B \) (see next slides).
- In all SAT competitions so far, DPLL-based procedures have shown the best performance.
Horn Clauses constitute an important special case, since they require only polynomial runtime of DPLL.

**Definition:** A Horn clause is a clause with maximally one positive literal. E.g., \( \neg A_1 \lor \ldots \lor \neg A_n \lor B \) or \( \neg A_1 \lor \ldots \lor \neg A_n \) \((n = 0 \text{ is permitted})\).

Equivalent representation: \( \neg A_1 \lor \ldots \lor \neg A_n \lor B \iff \bigwedge_i A_i \Rightarrow B \)

→ Basis of logic programming (e.g., PROLOG)
Note:

1. The simplifications in DPLL on Horn clauses always generate Horn clauses.

2. If the first sequence of applications of the unit propagation rule in DPLL does not lead to termination, a set of Horn clauses without unit clauses is generated.

3. A set of Horn clauses without unit clauses and without the empty clause is satisfiable, since:
   - All clauses have at least one negative literal (since all non-unit clauses have at least two literals, where at most one can be positive (Def. Horn))
   - Assigning false to all variables satisfies formula
4. It follows from 3.:
   a. every time the splitting rule is applied, the current formula is satisfiable
   b. every time, when the wrong decision (= assignment in the splitting rule) is made, this will be immediately detected (e.g., only through unit propagation steps and the derivation of the empty clause).

4. Therefore, the search trees for \( n \) variables can only contain a maximum of \( n \) nodes, in which the splitting rule is applied (and the tree branches).

4. Therefore, the size of the search tree is only polynomial in \( n \) and therefore the running time is also polynomial.
We know that SAT is NP-complete, i.e., in the worst case, it takes exponential time.

This is clearly also true for the DPLL-procedure. → Couldn’t we do better in the average case?

For CNF-formulae, in which the probability for a positive appearance, negative appearance and non-appearance in a clause is $1/3$, DPLL needs on average quadratic time (Goldberg 79)! → The probability that these formulae are satisfiable is, however, very high.
Conversely, we can, of course, try to identify hard to solve problem instances.

Cheeseman et al. (IJCAI-91) came up with the following plausible conjecture:

All NP-complete problems have at least one order parameter and the hard to solve problems are around a critical value of this order parameter. This critical value (a phase transition) separates one region from another, such as over-constrained and under-constrained regions of the problem space.

Confirmation for graph coloring and Hamilton path . . . , later also for other NP-complete problems.
Phase Transitions with 3-SAT

Constant clause length model (Mitchell et al., AAAI-92): Clause length $k$ is given. Choose variables for every clause $k$ and use the complement with probability 0.5 for each variable.

Phase transition for 3-SAT with a clause/variable ratio of approx. 4.3:
Empirical Difficulty

The Davis-Putnam (DPLL) Procedure shows extreme runtime peaks at the phase transition:

Note: Hard instances can exist even in the regions of the more easily satisfiable/unsatisfiable instances!
Notes on the Phase Transition

- When the probability of a solution is close to 1 (under-constrained), there are many solutions, and the first search path of a backtracking search is usually successful.

- If the probability of a solution is close to 0 (over-constrained), this fact can usually be determined early in the search.

- In the phase transition stage, there are many near successes ("close, but no cigar")
  → (limited) possibility of predicting the difficulty of finding a solution based on the parameters
  → (search intensive) benchmark problems are located in the phase region (but they have a special structure)
In many cases, we are interested in finding a satisfying assignment of variables (example CSP), and we can sacrifice completeness if we can “solve” much large instances this way.

Standard process for optimization problems: **Local Search**

- Based on a (random) configuration
- Through local modifications, we hope to produce better configurations
  
  → Main problem: **local maxima**
Dealing with Local Maxima

As a measure of the value of a configuration in a logical problem, we could use the number of satisfied constraints/clauses.

But local search seems inappropriate, considering we want to find a global maximum (all constraints/clauses satisfied).

By restarting and/or injecting noise, we can often escape local maxima.

Actually: Local search performs very well for finding satisfying assignments of CNF formulae (even without injecting noise).
**Procedure GSAT**

**INPUT**: a set of clauses $\alpha$, Max-Flips, and Max-Tries

**OUTPUT**: a satisfying truth assignment of $\alpha$, if found

begin

for $i := 1$ to Max-Tries

$T :=$ a randomly-generated truth assignment

for $j := 1$ to Max-Flips

if $T$ satisfies $\alpha$ then return $T$

$v :=$ a propositional variable such that a change in its truth assignment gives the largest increase in the number of clauses of $\alpha$ that are satisfied by $T$

$T := T$ with the truth assignment of $v$ reversed

end for

end for

return “no satisfying assignment found”

end
The Search Behavior of GSAT

- In contrast to normal local search methods, we must also allow sideways movements!
- Most time is spent searching on plateaus.
SAT competitions since beginning of the 90s

Current SAT competitions (http://www.satcompetition.org/):
  In 2010:
    - Largest “industrial” instances: > 1,000,000 literals

Complete solvers are as good as randomized ones on handcrafted and industrial problem
Concluding Remarks

- DPLL-based SAT solvers prevail:
  - Very efficient implementation techniques
  - Good branching heuristics
  - Clause learning

- Incomplete randomized SAT-solvers
  - are good (in particular on random instances)
  - but there is no dramatic increase in size of what they can solve
  - parameters are difficult to adjust