Introduction to Mobile Robotics

Probabilistic Robotics

Wolfram Burgard
Probabilistic Robotics

Key idea:

Explicit representation of uncertainty
(using the calculus of probability theory)

- Perception = state estimation
- Action = utility optimization
Axioms of Probability Theory

$P(A)$ denotes probability that proposition $A$ is true.

- $0 \leq P(A) \leq 1$
- $P(True) = 1$  \hspace{3cm} $P(False) = 0$
- $P(A \lor B) = P(A) + P(B) - P(A \land B)$
A Closer Look at Axiom 3

\[ P(A \lor B) = P(A) + P(B) - P(A \land B) \]
Using the Axioms

\[ P(A \vee \neg A) = P(A) + P(\neg A) - P(A \land \neg A) \]
\[ P(True) = P(A) + P(\neg A) - P(False) \]
\[ 1 = P(A) + P(\neg A) - 0 \]
\[ P(\neg A) = 1 - P(A) \]
Discrete Random Variables

- $X$ denotes a random variable
- $X$ can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$
- $P(X=x_i)$ or $P(x_i)$ is the probability that the random variable $X$ takes on value $x_i$
- $P(\cdot)$ is called probability mass function

E.g. $P(\text{Room}) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$
Continuous Random Variables

- $X$ takes on values in the continuum.
- $p(X=x)$ or $p(x)$ is a probability density function

\[ P(X \in [a, b]) = \int_a^b p(x) \, dx \]

- E.g.
“Probability Sums up to One”

Discrete case

\[ \sum_{x} P(x) = 1 \]

Continuous case

\[ \int p(x) \, dx = 1 \]
Joint and Conditional Probability

- \( P(X=x \text{ and } Y=y) = P(x,y) \)

- If \( X \) and \( Y \) are independent then
  \[ P(x,y) = P(x) P(y) \]

- \( P(x \mid y) \) is the probability of \( x \) given \( y \)
  \[ P(x \mid y) = \frac{P(x,y)}{P(y)} \]
  \[ P(x,y) = P(x \mid y) P(y) \]

- If \( X \) and \( Y \) are independent then
  \[ P(x \mid y) = P(x) \]
Law of Total Probability

**Discrete case**

\[ P(x) = \sum_{y} P(x \mid y)P(y) \]

**Continuous case**

\[ p(x) = \int p(x \mid y)p(y) \, dy \]
Marginalization

Discrete case

\[ P(x) = \sum_{y} P(x, y) \]

Continuous case

\[ \rho(x) = \int \rho(x, y) \, dy \]
Bayes Formula

\[ P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x) \]

\[ P(x \mid y) = \frac{P(y \mid x)P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}} \]
Normalization

\[ P(x \mid y) = \frac{P(y \mid x) P(x)}{P(y)} = \eta P(y \mid x) P(x) \]

\[ \eta = P(y)^{-1} = \frac{1}{\sum_x P(y \mid x) P(x)} \]

**Algorithm:**

\[ \forall x : \text{aux}_{x \mid y} = P(y \mid x) P(x) \]

\[ \eta = \frac{1}{\sum_x \text{aux}_{x \mid y}} \]

\[ \forall x : P(x \mid y) = \eta \text{aux}_{x \mid y} \]
Bayes Rule with Background Knowledge

\[ P(x \mid y, z) = \frac{P(y \mid x, z) P(x \mid z)}{P(y \mid z)} \]
Conditional Independence

\[ P(x, y \mid z) = P(x \mid z)P(y \mid z) \]

- Equivalent to \[ P(x \mid z) = P(x \mid z, y) \]

and \[ P(y \mid z) = P(y \mid z, x) \]

- But this does not necessarily mean \[ P(x, y) = P(x)P(y) \]

(independence/marginal independence)
Simple Example of State Estimation

- Suppose a robot obtains measurement $z$
- What is $P(\text{open} \mid z)$?
Causal vs. Diagnostic Reasoning

- $P(\text{open} | z)$ is diagnostic
- $P(z | \text{open})$ is causal
- In some situations, causal knowledge is easier to obtain

Bayes rule allows us to use causal knowledge:

$$P(\text{open} | z) = \frac{P(z | \text{open})P(\text{open})}{P(z)}$$

**count frequencies!**
Example

- \( P(z|\text{open}) = 0.6 \quad \text{and} \quad P(z|\neg\text{open}) = 0.3 \)
- \( P(\text{open}) = P(\neg\text{open}) = 0.5 \)

\[
P(\text{open} | z) = \frac{P(z | \text{open}) P(\text{open})}{P(z | \text{open}) P(\text{open}) + P(z | \neg\text{open}) P(\neg\text{open})}
\]
\[
P(\text{open} | z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{0.3}{0.3 + 0.15} = 0.67
\]

- \( z \) raises the probability that the door is open
Combining Evidence

- Suppose our robot obtains another observation $z_2$
- How can we integrate this new information?
- More generally, how can we estimate $P(x \mid z_1, \ldots, z_n)$?
Recursive Bayesian Updating

\[ P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x, z_1, \ldots, z_{n-1}) P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})} \]

**Markov assumption:**

\( z_n \) is independent of \( z_1, \ldots, z_{n-1} \) if we know \( x \)

\[ P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x) P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \ldots, z_{n-1})} = \eta \; P(z_n \mid x) P(x \mid z_1, \ldots, z_{n-1}) \]

\[ = \eta_{1 \ldots n} \left[ \prod_{i=1}^{n} P(z_i \mid x) \right] P(x) \]
Example: Second Measurement

- \( P(z_2|\text{open}) = 0.25 \)
- \( P(z_2|\neg\text{open}) = 0.3 \)
- \( P(\text{open}|z_1) = 2/3 \)

\[
P(\text{open}|z_2, z_1) = \frac{P(z_2|\text{open}) \cdot P(\text{open}|z_1)}{P(z_2|\text{open}) \cdot P(\text{open}|z_1) + P(z_2|\neg\text{open}) \cdot P(\neg\text{open}|z_1)}
\]

\[
= \frac{1 \cdot 2}{4 \cdot 3} + \frac{1 \cdot 1}{6 \cdot 10} = \frac{6}{8} = 0.625
\]

- \( z_2 \) lowers the probability that the door is open
Actions

- Often the world is **dynamic** since
  - actions carried out by the robot,
  - actions carried out by other agents,
  - or just the **time** passing by
  - change the world

- How can we **incorporate** such actions?
Typical Actions

- The robot **turns its wheels** to move
- The robot **uses its manipulator** to grasp an object
- Plants grow over **time** ...

- Actions are **never carried out with absolute certainty**
- In contrast to measurements, **actions generally increase the uncertainty**
Modeling Actions

- To incorporate the outcome of an action $u$ into the current “belief”, we use the conditional pdf $P(x \mid u, x')$

- This term specifies the pdf that executing $u$ changes the state from $x'$ to $x$. 
Example: Closing the door
State Transitions

\[ P(x \mid u, x') \text{ for } u = \text{“close door”}: \]

If the door is open, the action “close door” succeeds in 90% of all cases.
Integrating the Outcome of Actions

Continuous case:

\[ P(x \mid u) = \int P(x \mid u, x') P(x' \mid \square) \, dx' \]

Discrete case:

\[ P(x \mid u) = \sum P(x \mid u, x') P(x' \mid \square) \]

We will make an independence assumption to get rid of the \( u \) in the second factor in the sum.
Example: The Resulting Belief

\[
P(\text{closed} \mid u) = \sum P(\text{closed} \mid u, x') P(x')
\]
\[
= P(\text{closed} \mid u, \text{open}) P(\text{open}) + P(\text{closed} \mid u, \text{closed}) P(\text{closed})
\]
\[
= \frac{9}{10} \cdot \frac{5}{8} + \frac{1}{1} \cdot \frac{3}{8} = \frac{15}{16}
\]

\[
P(\text{open} \mid u) = \sum P(\text{open} \mid u, x') P(x')
\]
\[
= P(\text{open} \mid u, \text{open}) P(\text{open}) + P(\text{open} \mid u, \text{closed}) P(\text{closed})
\]
\[
= \frac{1}{10} \cdot \frac{5}{8} + \frac{0}{1} \cdot \frac{3}{8} = \frac{1}{16}
\]
\[
= 1 - P(\text{closed} \mid u)
\]
Bayes Filters: Framework

- **Given:**
  - Stream of observations $z$ and action data $u$:
    $$d_t = \{u_1, z_1, \ldots, u_t, z_t\}$$
  - Sensor model $P(z | x)$
  - Action model $P(x | u, x')$
  - Prior probability of the system state $P(x)$

- **Wanted:**
  - Estimate of the state $X$ of a dynamical system
  - The posterior of the state is also called **Belief**:
    $$Bel(x_t) = P(x_t | u_1, z_1, \ldots, u_t, z_t)$$
Markov Assumption

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

\[
P(z_t | x_{0:t}, z_{1:t-1}, u_{1:t}) = P(z_t | x_t)
\]

\[
P(x_t | x_{1:t-1}, z_{1:t-1}, u_{1:t}) = P(x_t | x_{t-1}, u_t)
\]
Bayes Filters

\[ \text{Bel}(x_t) = P(x_t \mid u_1, z_1, \ldots, u_t, z_t) \]

Bayes

\[ = \eta \ P(z_t \mid x_t, u_1, z_1, \ldots, u_t) \ P(x_t \mid u_1, z_1, \ldots, u_t) \]

Markov

\[ = \eta \ P(z_t \mid x_t) \ P(x_t \mid u_1, z_1, \ldots, u_t) \]

Total prob.

\[ = \eta \ P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots, u_t, x_{t-1}) \ P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \, dx_{t-1} \]

Markov

\[ = \eta \ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \ P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \, dx_{t-1} \]

Markov

\[ = \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \ P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1}) \, dx_{t-1} \]

\[ = \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \ \text{Bel}(x_{t-1}) \, dx_{t-1} \]
Bayes Filter Algorithm

1. Algorithm **Bayes_filter**\( (\text{Bel}(x), d) \):
2. \( \eta = 0 \)
3. If \( d \) is a perceptual data item \( z \) then
4. For all \( x \) do
5. \( \text{Bel}'(x) = P(z | x) \text{Bel}(x) \)
6. \( \eta = \eta + \text{Bel}'(x) \)
7. For all \( x \) do
8. \( \text{Bel}'(x) = \eta^{-1} \text{Bel}'(x) \)
9. Else if \( d \) is an action data item \( u \) then
10. For all \( x \) do
11. \( \text{Bel}'(x) = \int P(x | u, x') \text{Bel}(x') dx' \)
12. Return \( \text{Bel}'(x) \)

\[
\text{Bel}(x_t) = \eta P(z_t | x_t) \int P(x_t | u_t, x_{t-1}) \text{Bel}(x_{t-1}) dx_{t-1}
\]
Bayes Filters are Familiar!

\[ Bel(x_t) = \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1} \]

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)
Probabilistic Localization
Probabilistic Localization

\[ \text{Bel}(x \mid z, u) = \alpha \ p(z \mid x) \int_{x'} p(x \mid u, x') \text{Bel}(x') \, dx' \]
Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.