Foundations of Artificial Intelligence

6. Constraint Satisfaction Problems

CSPs as Search Problems, Solving CSPs, Problem Structure

Joschka Boedecker and Wolfram Burgard and
Frank Hutter and Bernhard Nebel and Michael Tangermann

Albert-Ludwigs-Universität Freiburg

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A motivating example of CSP (here: graph coloring)

- FCC Spectrum Auction in 2017
  - Wireless frequency spectra: demand increases
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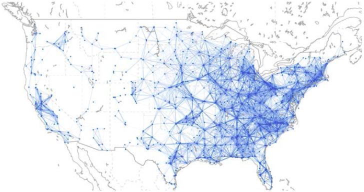
  - 2991 stations (nodes) &
    2.7 million interference constraints:
    stations in neighboring regions cannot use too similar frequencies
  - Need to check feasibility whenever an offer is made
  - More instances checkable: higher revenue



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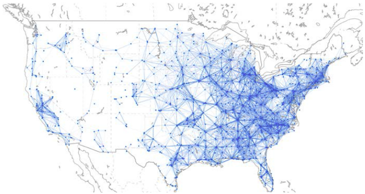


- Formulated as a CSP and solved with SAT solvers (improved by meta-algorithmics, see future lecture)
  - Improved ratio of instances solved from 73% to 99.6%
  - Net income for US government: $7 billion (used to pay down national debt)

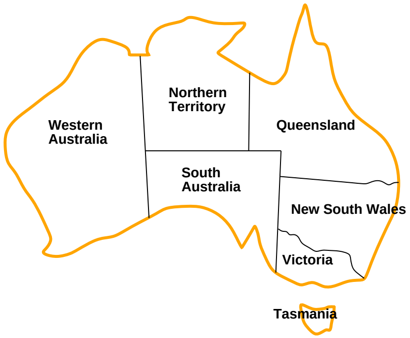
Contents

Lecture Overview

Constraint Satisfaction Problems

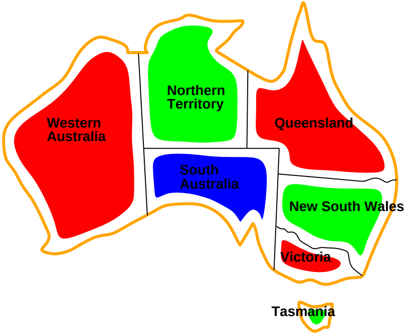
- A Constraint Satisfaction Problems (CSP) is given by
  - a set of variables $\{x_1, x_2, \ldots, x_n\}$,
  - an associated set of value domains $\{dom_1, dom_2, \ldots, dom_n\}$, and
  - a set of constraints. i.e., relations, over the variables.
  - An assignment of values to variables that satisfies all constraints is a solution of such a CSP.
- If CSPs are viewed as search problems, states are explicitly represented as variable assignments. CSP search algorithms take advantage of this structure.
- The main idea is to exploit the constraints to eliminate large portions of search space.
- *Formal representation language* with associated general inference algorithms

Example: Map-Coloring



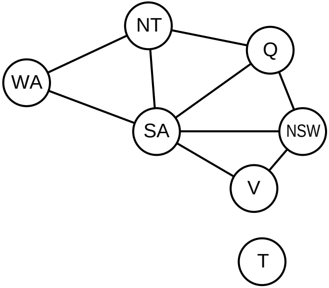
- Variables: WA, NT, SA, Q, NSW, V, T
- Values: $\{red, green, blue\}$
- Constraints: adjacent regions must have different colors, e.g., $NSW \neq V$

One Solution



- Solution assignment:
  - $\{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green\}$

Constraint Graph



- a constraint graph can be used to visualize binary constraints
- for higher order constraints, hyper-graph representations might be used
- Nodes = variables, arcs = constraints

Variations

- Binary, ternary, or even higher arity (e.g., ALL_DIFFERENT)
- Finite domains (d values) $\rightarrow d^n$ possible variable assignments
- Infinite domains (reals, integers)
  - *linear constraints (each variable occurs only in linear form)*: solvable (in P if real)
  - *nonlinear constraints*: unsolvable

Applications

- Timetabling (classes, rooms, times)
- Configuration (hardware, cars, . . .)
- Nurse rostering
- Scheduling (sports, etc)
- Sudoku
- . . .

Lecture Overview

Backtracking Search over Assignments

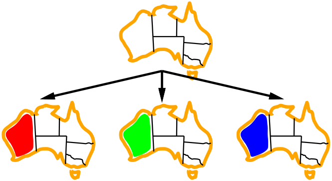
- Assign values to variables step by step (order does not matter)
- Consider only one variable per search node!
- DFS with single-variable assignments is called backtracking search

Algorithm

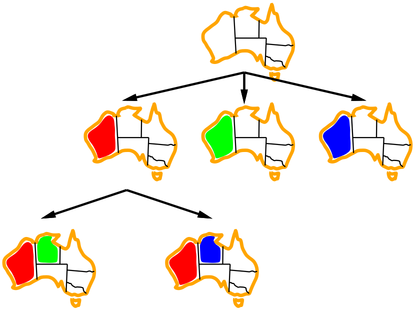
```
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
  return BACKTRACK({ }, csp)

function BACKTRACK(assignment, csp) returns a solution, or failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
     if value is consistent with assignment  then
        add {var = value} to assignment
        inferences ← INFERENCE(csp, var, value)
        if inferences ≠ failure then
           add inferences to assignment
           result ← BACKTRACK(assignment, csp)
           if result ≠ failure then
              return result
     remove {var = value} and inferences from assignment
  return failure
```

Example (1)

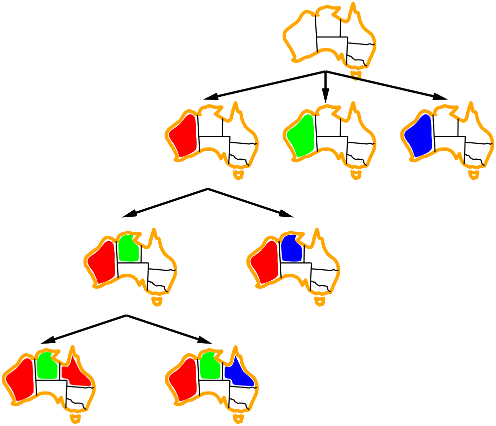
Example (2)



Example (3)



Example (4)



Lecture Overview

Improving Efficiency: CSP Heuristics & Pruning Techniques

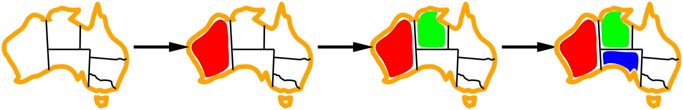
- Variable ordering: Which one to assign first?
- Value ordering: Which value to try first?
- Try to detect failures early on
- Try to exploit problem structure

→ Note: all this is not problem-specific!

Variable Ordering: Most constrained first

- Most constrained variable:
  - choose the variable with the fewest remaining legal values

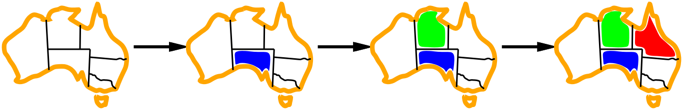
$\rightarrow$ reduces branching factor!



Variable Ordering: Most Constraining Variable First

- Break ties among variables with the same number of remaining legal values:
  - choose variable with the most constraints on remaining unassigned variables

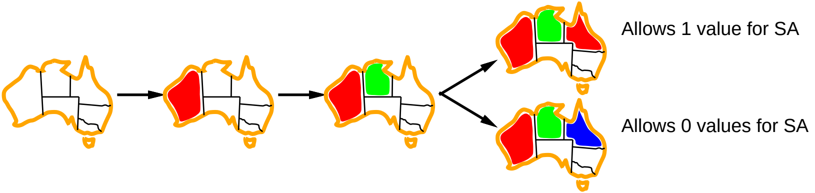
→ reduces branching factor in the next steps



Value Ordering: Least Constraining Value First

- Given a variable,
  - choose first a value that rules out the fewest values in the remaining unassigned variables

→ We want to find an assignment that satisfies the constraints (of course, this does not help if the given problem is unsatisfiable.)



Rule out Failures early on: Forward Checking

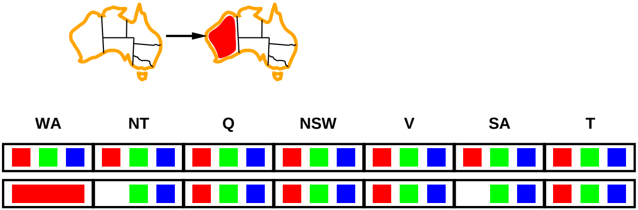
- Whenever a value is assigned to a variable, values that are now illegal for other variables are removed
- Implements what the ordering heuristics implicitly compute
- $WA =$ *red*, then NT cannot become *red*
- If all values are removed for one variable, we can stop!

Forward Checking (1)

- Keep track of remaining values
- Stop if all have been removed

Forward Checking (2)

- Keep track of remaining values
- Stop if all have been removed



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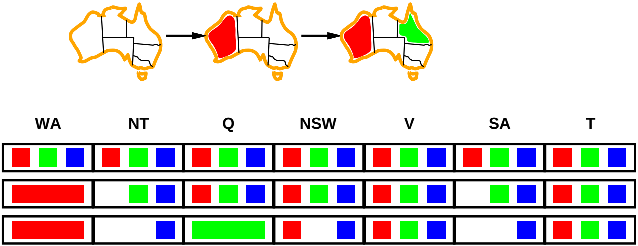

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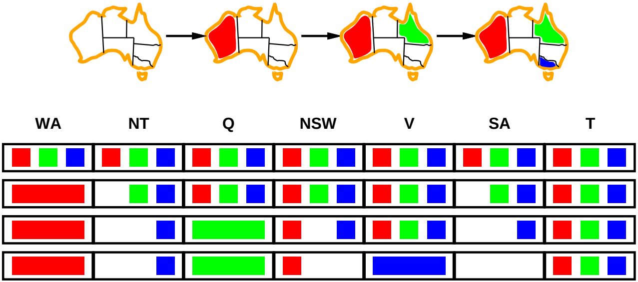
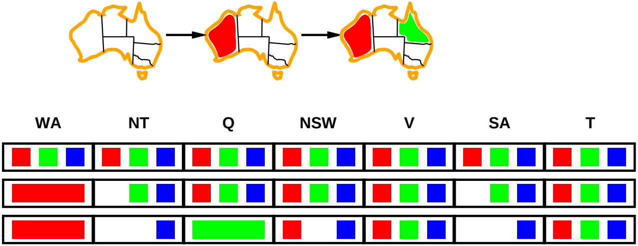


Lecture Overview

Forward Checking:
Sometimes it Misses Something

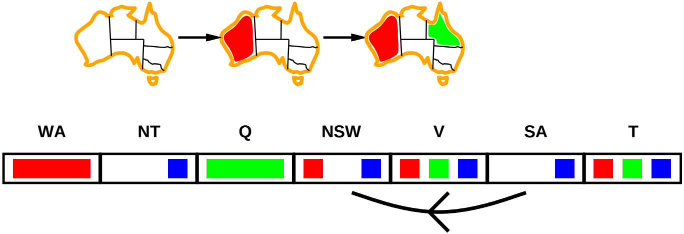
- Forward Checking propagates information from assigned to unassigned variables
- However, there is no propagation between unassigned variables



Arc Consistency

- A directed arc $X \rightarrow Y$ is "consistent" iff
  - for every value x of X, there exists a value y of Y, such that (x, y) satisfies the constraint between X and Y
- Remove values from the domain of X to enforce arc-consistency
- Arc consistency detects failures earlier
- Can be used as preprocessing technique or as a propagation step during backtracking

Arc Consistency Example



AC-3 Algorithm

```
function AC-3( csp) returns false if an inconsistency is found and true otherwise
  inputs: csp, a binary CSP with components (X, D, C)
  local variables: queue, a queue of arcs, initially all the arcs in csp

  while queue is not empty do
    (X_i, X_j) ← REMOVE-FIRST(queue)
    if REVISE(csp, X_i, X_j) then
      if size of D_i = 0 then return false
      for each X_k in X_i.NEIGHBORS - {X_j} do
        add (X_k, X_i) to queue
  return true
```

```
function REVISE( csp, X_i, X_j) returns true iff we revise the domain of X_i
  revised ← false
  for each x in D_i do
    if no value y in D_j allows (x,y) to satisfy the constraint between X_i and X_j then
      delete x from D_i
      revised ← true
  return revised
```

Properties of AC-3

- What is the computational complexity of AC-3?
  - Let n denote the number of nodes, and let d denote the maximal number of elements in a domain
  - Hint: what is the complexity of function REVISE, how often can it return true in the worst case, and how often is it thus called in the worst case?

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- AC-3 runs in $O(d^3 n^2)$ time
  - REVISE takes $O(d^2)$ (for each element $x \in D_i$, you need to check each element $y \in D_j$)
  - Each time REVISE returns true one element of X_i is eliminated; there are only max. d elements for each of the n variables
  - Each time REVISE returns true up to n constraints are added to the queue
  - Alltogether, in the worst case, REVISE can only be called a maximum of $O(n^2 d)$ times, each taking time $O(d^2)$

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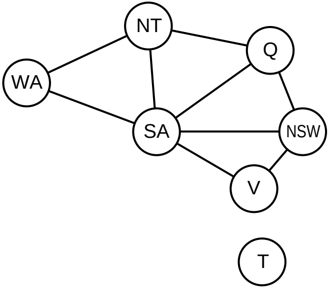
- Of course, AC-3 does not detect all inconsistencies (which is an NP-hard problem)

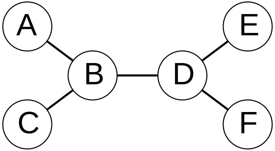
Lecture Overview

Problem Structure (1)



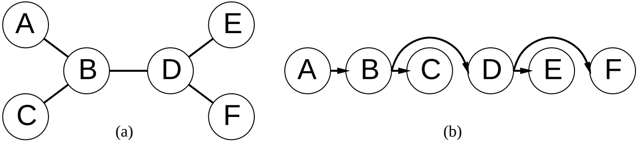
- This example CSP has two independent components
- Identifiable as connected components of constraint graph
- Can reduce the search space dramatically

Problem Structure (2): Tree-structured CSPs



- If the CSP graph is a tree, then it can be solved in $O(nd^2)$ (general CSPs need in the worst case $O(d^n)$).
- *Idea:* Pick root, order nodes, apply arc consistency from leaves to root, and assign values starting at root.

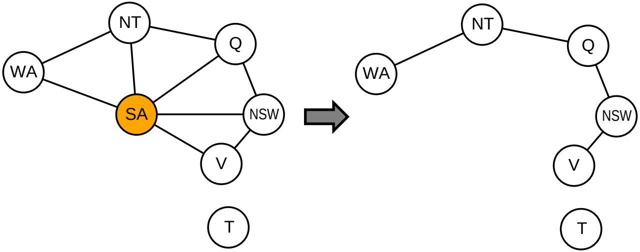
Problem Structure (2): Tree-structured CSPs



- Pick any variable as root; choose an ordering such that each variable appears after its parent in the tree.
- Apply arc-consistency to (x_i, x_k) when x_i is the parent of x_k for all $k = n$ down to 2 (any tree with n nodes has $n - 1$ arcs and per arc d^2 comparisons are needed, which results in a complexity of $O(n\, d^2)$).
- Now we can start at x_1 assigning values from the remaining domains without creating any conflict in one sweep through the tree!
- This algorithm is linear in n.

Problem Structure (3): Almost Tree-structured

Idea: Reduce the graph structure to a tree by fixing values in a reasonably chosen subset

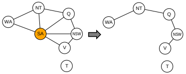


Instantiate a variable and prune values in neighboring variables is called Conditioning

Problem Structure (4): Almost Tree-structured

Algorithm Cutset Conditioning:

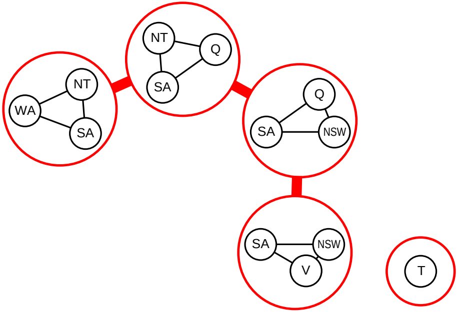
1. Choose a subset S of the CSPs variables such that the constraint graph becomes a tree after removal of S. The set S is called a cycle cutset.
2. For each possible assignment of variables in S that satisfies all constraints on S
   1. remove from the domains of the remaining variables any values that are inconsistent with the assignments for S, and
   2. if the remaining CSP has a solution, return it together with the assignment for S



Note: Finding the smallest cycle cutset is NP hard, but several efficient approximation algorithms are known.

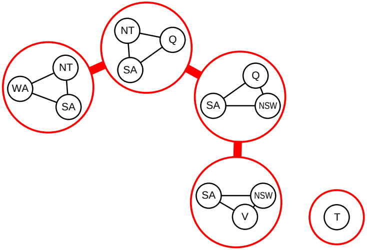
Another Method: Tree Decomposition (1)

- Decompose the problem into a set of connected sub-problems, where two sub-problems are connected when they share a constraint
- Solve the sub-problems independently and then combine the solutions



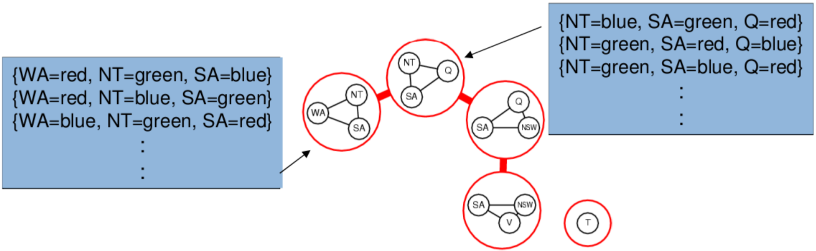
Another Method: Tree Decomposition (2)

- A tree decomposition must satisfy the following conditions:
  - Every variable of the original problem appears in at least one sub-problem
  - Every constraint appears in at least one sub-problem
  - If a variable appears in two sub-problems, it must appear in all sub-problems on the path between the two sub-problems
  - The connections form a tree



Another Method: Tree Decomposition (3)

- Consider sub-problems as new mega-variables, which have values defined by the solutions to the sub-problems
- Use technique for tree-structured CSP to find an overall solution (constraint is to have identical values for the same variable)



Tree Width

- The aim is to make the subproblems as small as possible. The tree width w of a tree decomposition is the size of largest sub-problem minus 1
- Tree width of a graph is minimal tree width over all possible tree decompositions
- If a graph has tree width w and we know a tree decomposition with that width, we can solve the problem in $O(nd^{w+1})$
- Unfortunately, finding a tree decomposition with minimal tree width is NP-hard. However, there are heuristic methods that work well in practice.

Summary

- CSPs are a special kind of search problem:
  - states are value assignments
  - goal test is defined by constraints
- Backtracking = DFS with one variable assigned per node. Other intelligent backtracking techniques possible
- Variable/value ordering heuristics can help dramatically
- Constraint propagation prunes the search space
- Tree structure of CSP graph simplifies problem significantly
- Cutset conditioning and tree decomposition are two ways to transform part of the problem into a tree
- CSPs can also be solved using local search