Question 1 (Finite Automata, 8 + 6 points)

(a) Give a regular expression for each of the following languages:

(i) all strings over \{0, 1\} that are at least three symbols long and have a 0 at their resp. 3rd positions

(ii) all strings over \{0, 1\} that have odd length, if starting with a 0, and even length otherwise

(iii) all strings over \{a, b\} that contain the substrings aa or bab

(iv) all strings over \{a, b\} that do not contain the substring ba

Solution: Setting \(\Sigma := \{0, 1\}\) or \(\Sigma := \{a, b\}\) according to the context, some possible solutions are

(i) \(\Sigma \Sigma 0 \Sigma^*\).

(ii) \((0 \cup \epsilon)(\Sigma \Sigma)^*\).

(iii) \(\Sigma^*(aa \cup bab) \Sigma^*\).

(iv) If a string on \{a, b\} does not contain ba as a substring, it means that the sequence of symbols is not decreasing (wrt the lexicographic order). Thus a solution is \(a^*b^*\).

(b) Draw a DFA equivalent to each of the following regular expressions:

(i) \(a(a \cup b)^*b\).

Solution: a possible DFA is:

(ii) \((ab)^*\)

Solution: below an example of DFA that accepts such language
**Question 2** (Regular languages, 14 points)

Let $\Sigma = \{a, b\}$. Use the pumping lemma to prove that:

$$L = \{a^n b^{2n} a^{3n} \mid n \geq 0\}$$

is not regular.

Any other proof techniques will not receive any points.

**Solution:** let’s assume that $L$ is a regular language. According to the Pumping Lemma, there must exist an integer $p > 0$ so that for any word $w \in L$ with $|w| \geq p$, there exists a decomposition of $w$ in substrings $xyz$ (i.e. $w = xyz$) such that the following properties are satisfied:

- $|xy| \leq p$.
- $y \neq \epsilon$
- $xy^kz \in L$ for any $k \in \mathbb{N}_0$.

It is apparent that if we select $w = a^p b^{2p} a^{3p} \in L$, then $|w| \geq p$. Furthermore, if such $x, y, z$ exist, then $xy^0z = a^{p-|y|} b^{2p} a^{3p} \notin L$. This contradicts the Pumping Lemma.

**Question 3** (Context-free languages, 7+7 points)

(a) Give the state diagram of a PDA recognizing the language

$$A = \{a^i b^j \mid i > 0 \text{ and } j = i + 1\}.$$ 

**Solution:** it is easy to see that the following PDA accept the language $A$:

(b) Let $G = \langle \{S, X, Y, Z\}, \{a, b, c, d\}, R, S \rangle$ be the CFG with rules:

$$S \rightarrow XYZ$$
$$X \rightarrow Xa \mid b \mid \epsilon$$
$$Y \rightarrow b \mid c$$
$$Z \rightarrow cd$$

Specify a CFG $G_0$ in Chomsky Normal Form such that $L(G_0) = L(G)$.

**Solution:** we apply the standard procedure to convert CFG in Chomsky Normal Form. The procedure is listed below:

Remove the $\epsilon$-rules:

$$S \rightarrow XYZ \mid YZ$$
$$X \rightarrow Xa \mid b \mid a$$
$$Y \rightarrow b \mid c$$
$$Z \rightarrow cd$$
Remove \( [X \to Xa] \) by introducing the auxiliary variable \( A \) and the rule \( [A \to a] \):

\[
S \to XYZ \mid YZ \\
X \to XA \mid b \mid a \\
Y \to b \mid c \\
A \to a \\
Z \to cd
\]

Add auxiliary variables \( U, C, D \) and related rules to complete the CNF.

\[
S \to XU \mid YZ \\
U \to YZ \\
X \to XA \mid b \mid a \\
Z \to CD \\
A \to a \\
C \to c \\
D \to d \\
Y \to b \mid c
\]

**Question 4** (NP-completeness, 7 + 7 points)

Let \( G := (V, E) \) be an undirected graph. A vertex cover of \( G \) is a vertex set \( C \subseteq V \) such that for all \( \langle u, v \rangle \in E \), \( u \in C \) or \( v \in C \).

Let \( S := (S, C) \) be a subset collection, i.e., \( S \) is a finite set and \( C = \{C_1, \ldots, C_n\} \) where \( C_i \subseteq S \) for all \( i \in \{1, \ldots, n\} \). A hitting set of \( S \) is a subset \( H \subseteq S \) such that \( H \cap C_i \neq \emptyset \) for all \( i \in \{1, \ldots, n\} \).

The **VertexCover** and **HittingSet** decision problems are defined as:

- **VertexCover** = \( \{\langle G, n \rangle \mid G \text{ is a graph which has a vertex cover of size at most } n \in \mathbb{N}_1\} \)
- **HittingSet** = \( \{\langle S, m \rangle \mid S \text{ is a subset collection with a hitting set of size at most } m \in \mathbb{N}_1\} \)

(a) Prove that **VertexCover** \( \leq_p \) **HittingSet**.

(b) Prove that **HittingSet** is NP-complete. (You may use your result from part (a) and that it is known that **VertexCover** is NP-complete.)

**Solution:**

(a) Our goal is to prove that there exists a function \( f \) which convert the input of **VertexCover** into the input of **HittingSet** and that can be computed in polynomial time by a Turing Machine. Formally, we need to provide a function \( f \) so that \( \langle G, n \rangle \in \text{VertexCover} \iff f(\langle G, n \rangle) \in \text{HittingSet} \) and \( f \in O(|\langle G, n \rangle|^k) \) for some integer \( k > 0 \).

To do so, let \( E := \{(u, v) \mid (u, v) \in E\} \). Then it easy to see that

\( C \) is a vertex cover of \( G \) of size at most \( n \) \iff \( C \) is an hitting set of \( (V, E) \) of size at most \( n \)

Let now \( f \) be the function defined so that \( f(\langle G, n \rangle) = \langle E, n \rangle \), it is easy to see from the observation above that \( \langle G, n \rangle \in \text{VertexCover} \iff f(\langle G, n \rangle) \in \text{HittingSet} \) and \( f \in O(|\langle G, n \rangle|^2) \).
(b) We have to show that

**HittingSet is NP-hard:** that is, $X \leq_p \text{HittingSet} \ \forall X \in \text{NP}$.

This can easily proven just by observing that **VertexCover** is NP-complete, that is, $X \leq_p \text{VertexCover}$ for all $X \in \text{NP}$. Thus, $X \leq_p \text{VertexCover} \leq_p \text{HittingSet} \ \forall X \in \text{NP}$.

**HittingSet $\in$ NP:** We can apply guess-and-check to solve **HittingSet**. Indeed, we can test whether a subset $H$ is an hitting set of size at most $m$ for $S$ by testing the non-emptiness of the intersection $H \cap C_i$ for every set $C_i \in C$. This can surely be done in $O(|H| \sum_{i=1}^{|C|} |C_i|)$. Observing that $|C_i| \leq |S|$, $|C| \leq m$ and $|H| \leq |S|$, then it easy to see that such test can be performed in $O(m|S|^2)$ computations. That is, a Turing machine $M$ can run such test in $O(|\langle S, m \rangle|^3)$. Thus we can create a NTM $M'$ that on input $\langle S, m \rangle$ choose non-deterministically a set $H \subseteq S$ and test whether $H$ is an hitting set for $\langle S, m \rangle$. $M'$ solves **HittingSet** in polynomial time.

**Question 5** (Decidability, 4 + 10 points)

Consider the problem of testing whether a given single-tape Turing machine ever writes a blank symbol over a non-blank symbol during the course of its computation, for any input string.

(a) Formulate this problem as a language.

(b) Show that the problem is undecidable.

**Solution:**

(a) The language can be described as

$L = \{ \langle M, w \rangle \mid M \text{ is a TM that writes a blank symbol over a non-blank one } \ldots \}$.

(b) To show that such language is undecidable, we can argument as follows. Let suppose that $L$ is decidable and let $D$ be a decider for $L$. We can define a Turing Machine $D$ as follows:

$D = \text{"On input } \langle M, w \rangle \text{"

1. Create a Turing Machine $M'$ so that:
   1.a Whenever $M$ writes a blank symbol, it writes a non-blank symbol $\gamma$.
   1.b Apply the transition defined by blank symbols whenever $\gamma$ is read.
   1.c Before accepting, we write $\gamma$ and then we overwrite a blank.

2. If $D(\langle M', w \rangle)$ accepts, accept. reject otherwise."}

It is easy to see that $M'$ never writes a blank symbol unless $M'$ accepts the input $w$. That is, $D$ is a decider for $A_{TM}$ which is clearly a contradiction.

**Question 6** (Propositional Logic, 5 + 9 points)

(a) Resolution is not a complete proof method. However, the contradiction theorem can be used to obtain a sound and complete method based on resolution for answering queries of the form “Does $\text{KB} \models \varphi$?”. Describe how this is done in general, i.e., to which set of clauses the resolution method is applied, and which outcome of the resolution method means that $\text{KB} \models \varphi$.

You may assume that $\text{KB}$ is given as a set of clauses and $\varphi$ as a conjunction of literals.
(b) Use the method described in part (a) to prove $KB \models P \land R$ for

$$KB = \{ P \lor \neg Q, \ P \lor Q \lor \neg R, \ P \lor R, \ Q \lor S, \ R, \ \neg R \lor S \}.$$ 

**Solution:**

(a) Discussed in class.

(b) Using contradiction theorem we have $KB \models P \lor R$ if and only if $KB' := KB \cup \{ \neg (P \land R) \} \models \bot$.

We first convert $KB'$ into a clause set:

<table>
<thead>
<tr>
<th>Formula (and equivalences)</th>
<th>Clauses</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P \lor \neg Q$</td>
<td>${P, \neg Q}$</td>
</tr>
<tr>
<td>$P \lor Q \lor \neg R$</td>
<td>${P, Q, \neg R}$</td>
</tr>
<tr>
<td>$P \lor R$</td>
<td>${P, R}$</td>
</tr>
<tr>
<td>$Q \lor S$</td>
<td>${Q, S}$</td>
</tr>
<tr>
<td>$R$</td>
<td>${R}$</td>
</tr>
<tr>
<td>$\neg R \lor S$</td>
<td>${\neg R, S}$</td>
</tr>
<tr>
<td>$\neg (P \land R) \equiv \neg P \lor \neg R$</td>
<td>${\neg P, \neg R}$</td>
</tr>
</tbody>
</table>

That is, the clause set is

$$\Delta := \{ \{P, \neg Q\}, \{P, Q, \neg R\}, \{P, R\}, \{Q, S\}, \{R\}, \{\neg R, S\}, \{\neg P, \neg R\} \}.$$ 

The following derivation turns into a contradiction:

- $C_1 = \{P, \neg Q\}$ from $\Delta$
- $C_2 = \{P, Q, \neg R\}$ from $\Delta$
- $C_3 = \{P, \neg R\}$ from $C_1$ and $C_2$
- $C_4 = \{R\}$ from $\Delta$
- $C_5 = \{P\}$ from $C_3$ and $C_4$
- $C_6 = \{\neg P, \neg R\}$ from $\Delta$
- $C_7 = \{\neg R\}$ from $C_6$ and $C_5$
- $C_8 = \Box$ from $C_4$ and $C_7$

**Question 7** (Propositional logic, 9 + 5 points)

(a) Which of the following formulae are satisfiable? Which ones are valid? Which ones are unsatisfiable? For formulas belonging to several of these categories, please list all of them.

For all satisfiable cases, also provide a satisfying truth assignment. For the questions about validity and unsatisfiability, you do not need to justify your answers.

(i) $(A \lor \neg B) \rightarrow (A \land C)$
(ii) $(A \leftrightarrow B) \land (B \leftrightarrow \neg A)$
(iii) $(A \land B) \lor (\neg A \land \neg B)$
(iv) $(A \leftrightarrow B) \land (B \rightarrow \neg A)$

**Solution:**

(i) Satisfiable ($A \leftarrow T, B \leftarrow T, C \leftarrow T$).
(ii) Unsatisfiable.
(iii) Satisfiable ($A \leftarrow T, B \leftarrow T$).
(iv) Unsatisfiable.

(b) Prove that

$$(A \land B) \rightarrow C \equiv A \rightarrow (B \rightarrow C)$$

by providing a sequence of logical equivalences that transforms the left-hand side into the right-hand side.

Solution: We can prove the equivalence as follows:

$$(A \land B) \rightarrow B \equiv A \rightarrow (B \rightarrow C)$$
$$\neg(A \land B) \lor C \equiv \neg A \lor (B \rightarrow C)$$
$$\neg A \lor \neg B \lor C \equiv \neg A \lor \neg B \lor C$$

**Question 8** (Example of multiple choice question)

In which of the following cases is the logical formula to the left a reasonable formalization of the natural-language sentence to the right?

- $\forall x \forall y(LivesIn(x, y) \land \neg EatsUp(x)) \rightarrow BadWeatherIn(y)$ “Whenever someone who lives in some place does not eat up, the weather in that place will be bad.”
- $\forall x \forall y(Friend(x, y) \land Friend(y, x))$ “Whenever A is a friend of B, B is a friend of A.”
- $\forall x \forall y(FatherOf(x, me) \land DaughterOf(y, x) \land Female(me)) \rightarrow (y = me)$ “If my father has a daughter and I am female, then that daughter is me.”
- $\exists x \forall y Father(x, y)$ “Everybody has at least one father.”
- $DaughterOf(me, Friend)$ “I am the daughter of my friend.”

Solution: 1.