

Robot Mapping

A Short Introduction to the Bayes Filter and Related Models

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State Estimation

- Estimate the state x of a system given observations z and controls u
- **Goal:**

$$p(x \mid z, u)$$

Recursive Bayes Filter 1

$$bel(x_t) = \underline{p(x_t \mid z_{1:t}, u_{1:t})}$$

Definition of the belief

Recursive Bayes Filter 2

$$\begin{aligned} \text{bel}(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\ &= \eta \underbrace{p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) p(x_t \mid z_{1:t-1}, u_{1:t})} \end{aligned}$$

Bayes' rule

Recursive Bayes Filter 3

$$\begin{aligned} \text{bel}(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\ &= \eta p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \eta \underline{p(z_t \mid x_t)} p(x_t \mid z_{1:t-1}, u_{1:t}) \end{aligned}$$

Markov assumption

Recursive Bayes Filter 4

$$\begin{aligned} \text{bel}(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\ &= \eta p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \eta p(z_t \mid x_t) p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \eta p(z_t \mid x_t) \int \underbrace{p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})}_{\underbrace{p(x_{t-1} \mid z_{1:t-1}, u_{1:t})} dx_{t-1}} \end{aligned}$$

Law of total probability

Recursive Bayes Filter 5

$$\begin{aligned} \text{bel}(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\ &= \eta p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \eta p(z_t \mid x_t) p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \eta p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\ &\quad p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) dx_{t-1} \\ &= \eta p(z_t \mid x_t) \int \underline{p(x_t \mid x_{t-1}, u_t)} p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) dx_{t-1} \end{aligned}$$

Markov assumption

Recursive Bayes Filter 6

$$\begin{aligned} \text{bel}(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\ &= \eta p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \eta p(z_t \mid x_t) p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \eta p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\ &\quad p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) dx_{t-1} \\ &= \eta p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t) p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) dx_{t-1} \\ &= \eta p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t) p(x_{t-1} \mid z_{1:t-1}, \underline{u_{1:t-1}}) dx_{t-1} \end{aligned}$$

Markov assumption

Recursive Bayes Filter 7

$$\begin{aligned} \text{bel}(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\ &= \eta p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \eta p(z_t \mid x_t) p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \eta p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\ &\quad p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) dx_{t-1} \\ &= \eta p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t) p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) dx_{t-1} \\ &= \eta p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t) p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1}) dx_{t-1} \\ &= \eta p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t) \underline{\text{bel}(x_{t-1})} dx_{t-1} \end{aligned}$$

Recursive term

Prediction and Correction Step

- Bayes filter can be written as a two step process
- **Prediction step**

$$\overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1}$$

- **Correction step**

$$bel(x_t) = \eta p(z_t | x_t) \overline{bel}(x_t)$$

Motion and Observation Model

- Prediction step

$$\overline{bel}(x_t) = \int \underline{p(x_t | u_t, x_{t-1})} bel(x_{t-1}) dx_{t-1}$$

motion model

- Correction step

$$bel(x_t) = \eta \underline{p(z_t | x_t)} \overline{bel}(x_t)$$

sensor or observation model

Different Realizations

- The Bayes filter is a **framework** for recursive state estimation
- There are **different realizations**
- **Different properties**
 - Linear vs. non-linear models for motion and observation models
 - Gaussian distributions only?
 - Parametric vs. non-parametric filters
 - ...

In this Course

- **Kalman filter & friends**
 - Gaussians
 - Linear or linearized models

- **Particle filter**
 - Non-parametric
 - Arbitrary models (sampling required)

Motion Model

$$\overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1}$$

Probabilistic Motion Models

- Specifies a posterior probability that action u carries the robot from x to x' .

$$p(x_t \mid u_t, x_{t-1})$$

Typical Motion Models

- In practice, one often finds two types of motion models:
 - **Odometry-based**
 - **Velocity-based**
- Odometry-based models for systems that are equipped with wheel encoders
- Velocity-based when no wheel encoders are available

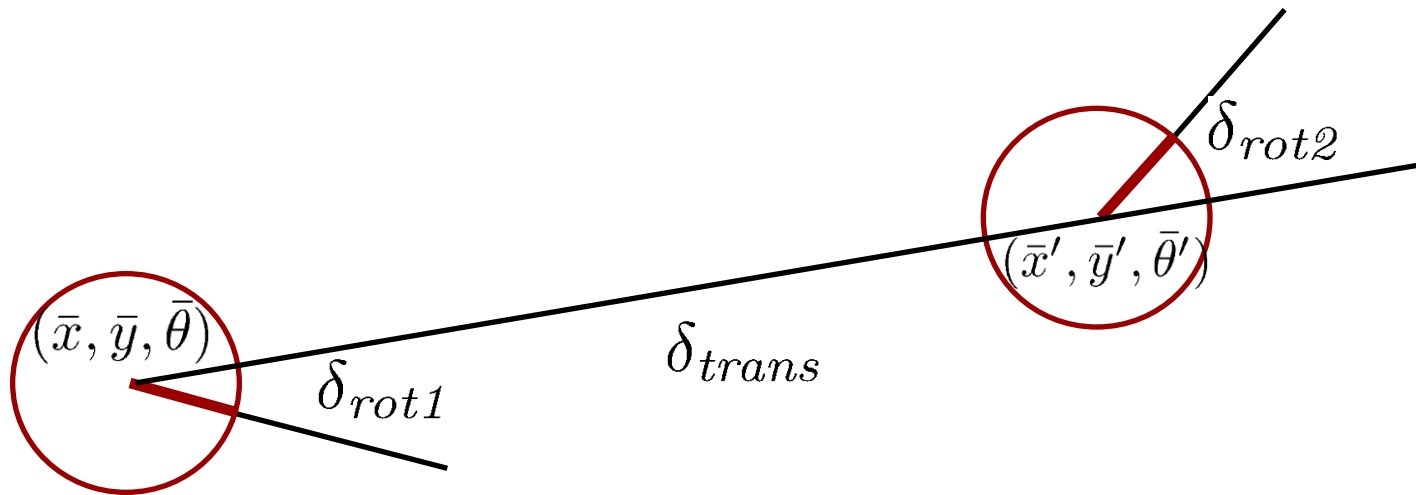
Odometry Model

- Robot moves from $(\bar{x}, \bar{y}, \bar{\theta})$ to $(\bar{x}', \bar{y}', \bar{\theta}')$
- Odometry information $u = (\delta_{rot1}, \delta_{trans}, \delta_{rot2})$

$$\delta_{trans} = \sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2}$$

$$\delta_{rot1} = \text{atan2}(\bar{y}' - \bar{y}, \bar{x}' - \bar{x}) - \bar{\theta}$$

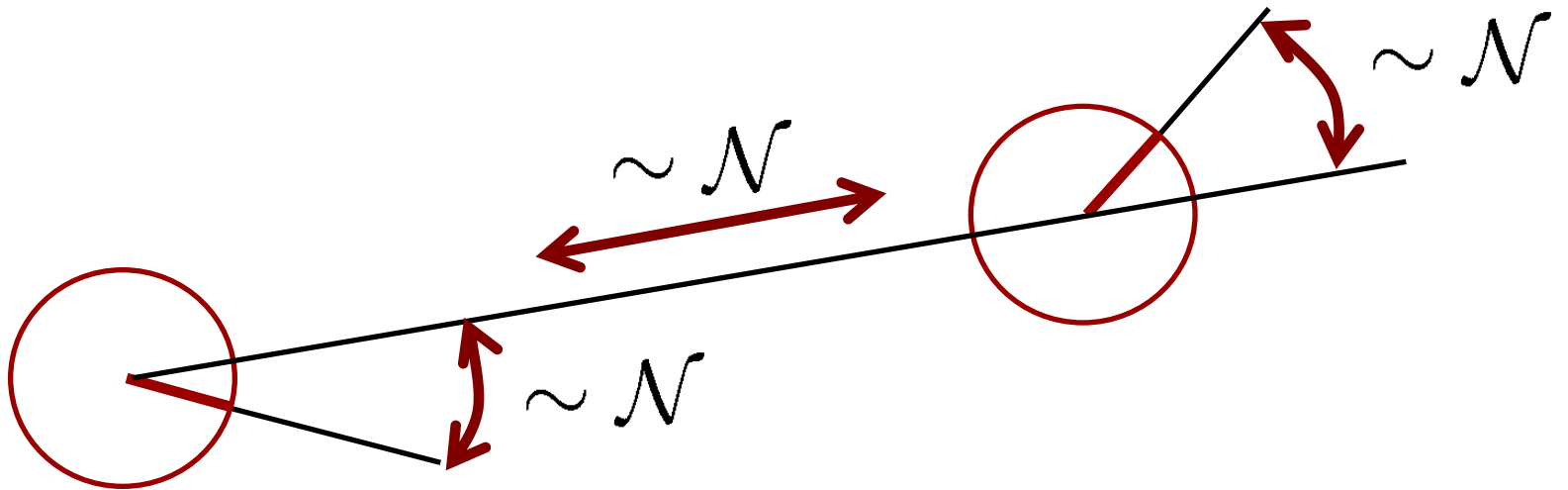
$$\delta_{rot2} = \bar{\theta}' - \bar{\theta} - \delta_{rot1}$$



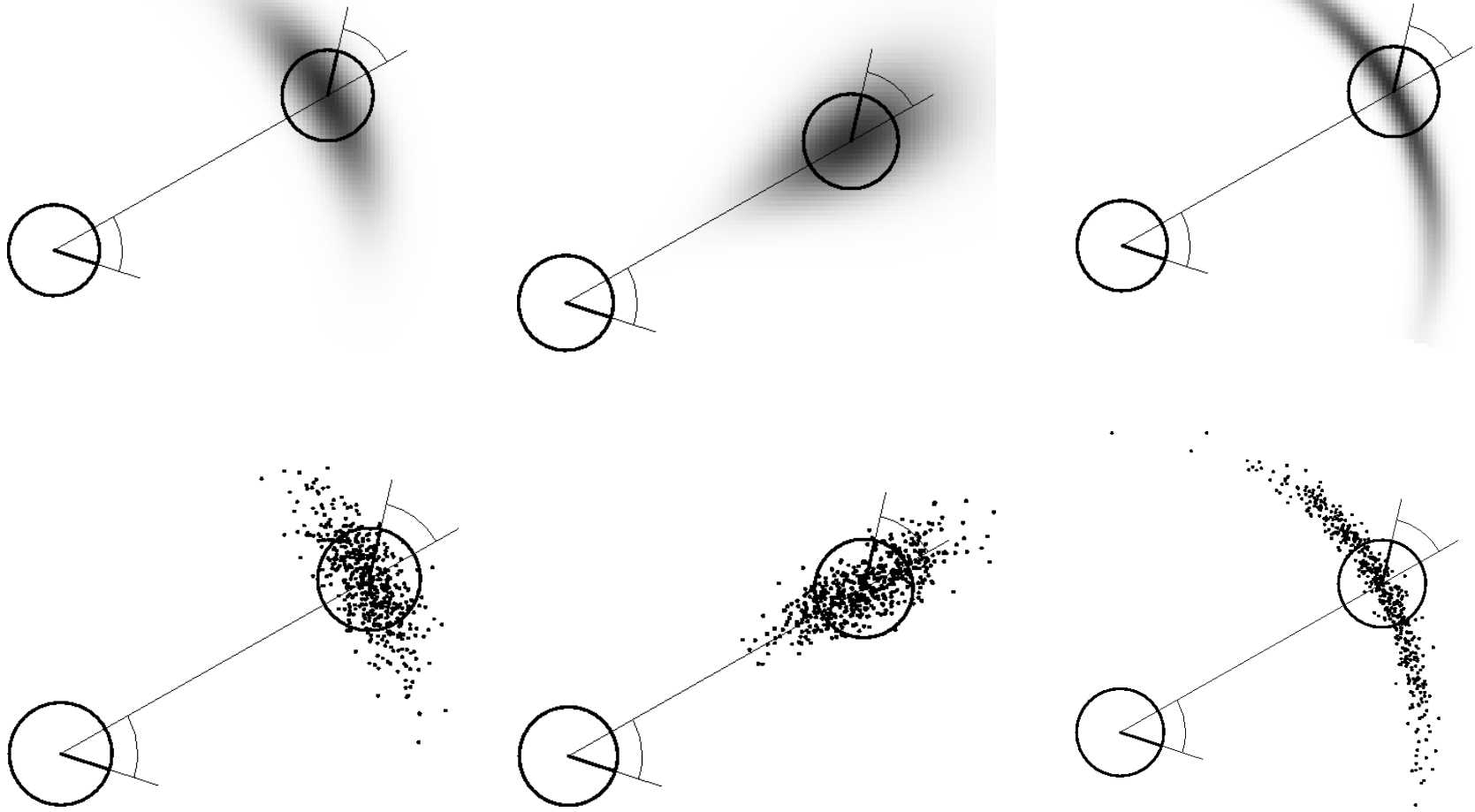
Probability Distribution

- Noise in odometry $u = (\delta_{rot1}, \delta_{trans}, \delta_{rot2})$
- Example: Gaussian noise

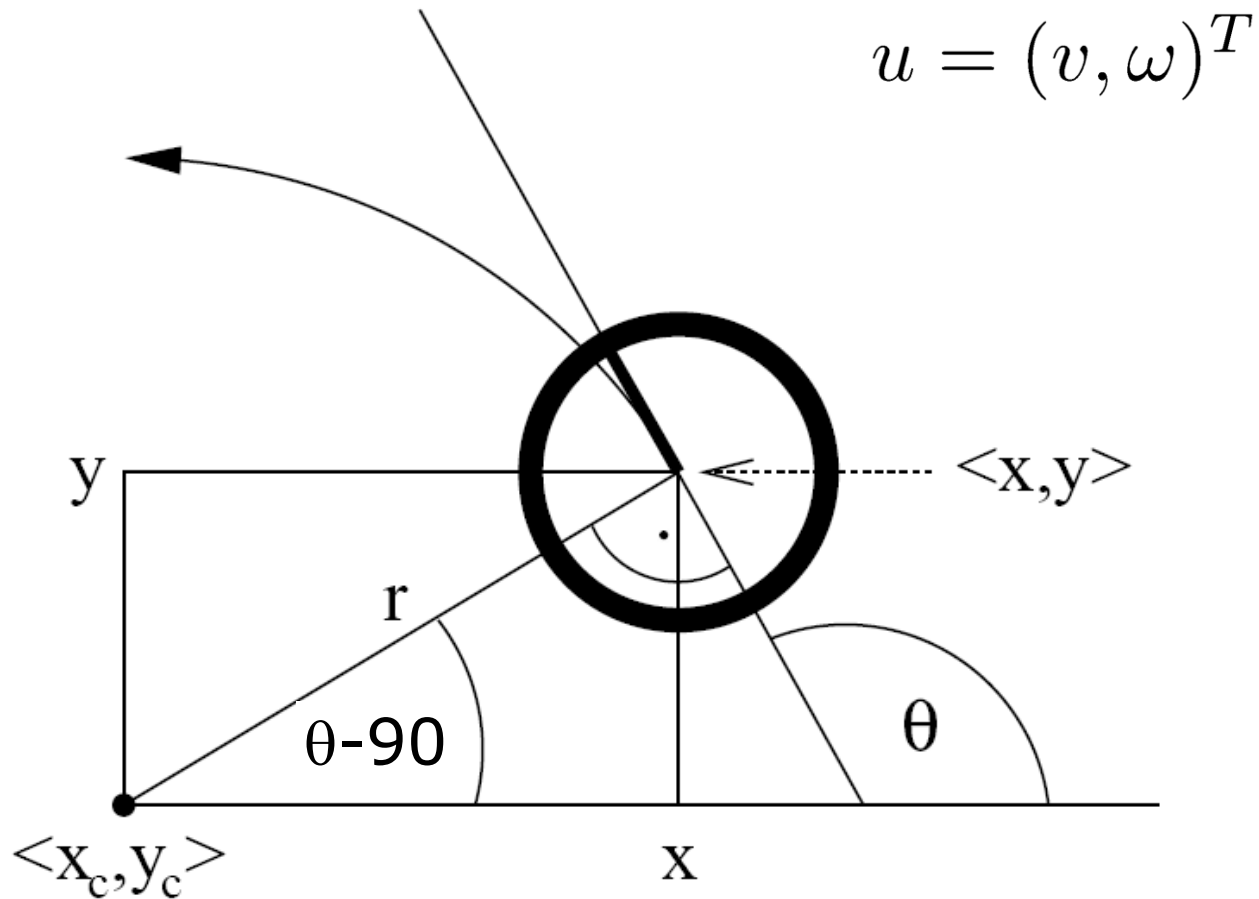
$$u \sim \mathcal{N}(0, \Sigma)$$



Examples (Odometry-Based)



Velocity-Based Model



Motion Equation

- Robot moves from (x, y, θ) to (x', y', θ')
- Velocity information $u = (v, \omega)$

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v}{\omega} \sin \theta + \frac{v}{\omega} \sin(\theta + \omega \Delta t) \\ \frac{v}{\omega} \cos \theta - \frac{v}{\omega} \cos(\theta + \omega \Delta t) \\ \omega \Delta t \end{pmatrix}$$

Problem of the Velocity-Based Model

- Robot moves on a circle
- The circle constrains the final orientation
- **Fix:** introduce an additional noise term on the final orientation

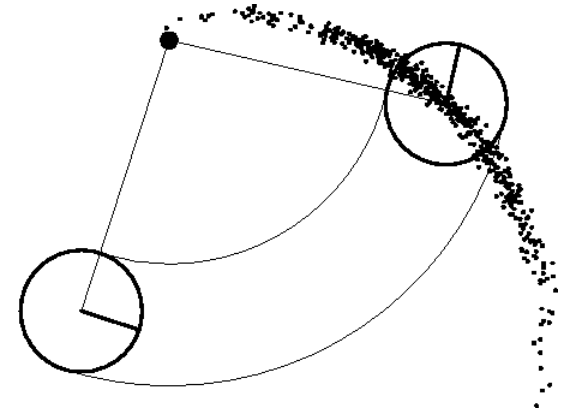
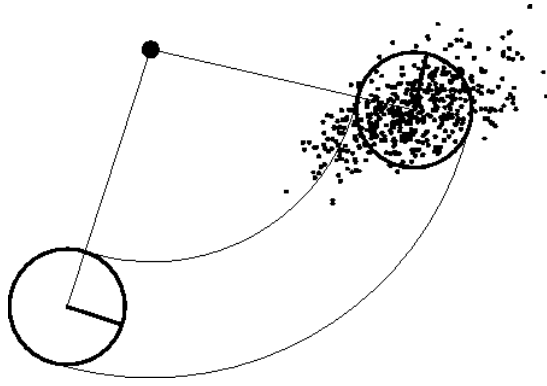
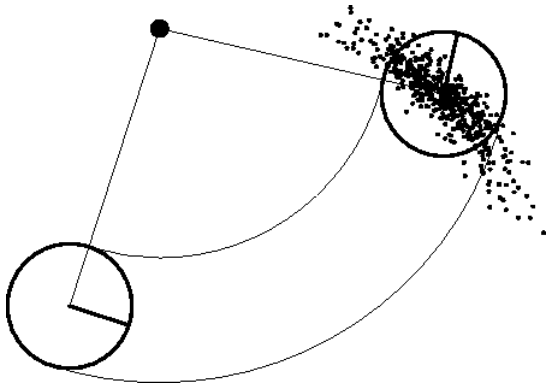
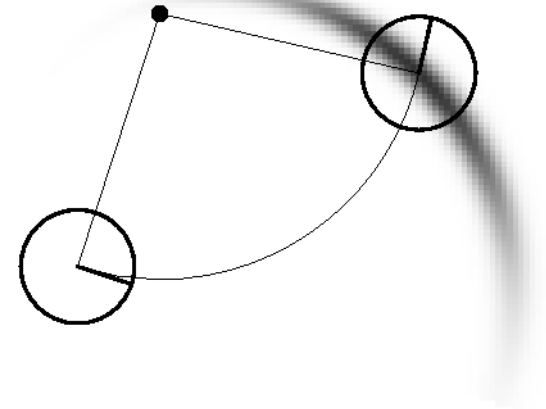
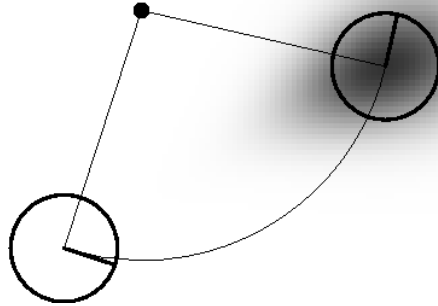
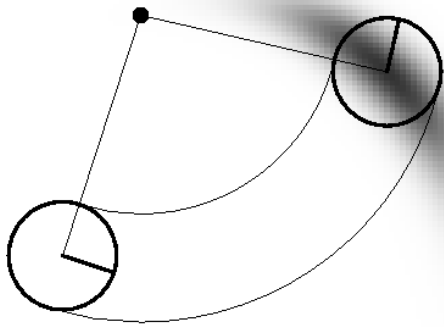
Motion Including 3rd Parameter

$$\begin{pmatrix} x' \\ y' \\ \theta' \end{pmatrix} = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} + \begin{pmatrix} -\frac{v}{\omega} \sin \theta + \frac{v}{\omega} \sin(\theta + \omega \Delta t) \\ \frac{v}{\omega} \cos \theta - \frac{v}{\omega} \cos(\theta + \omega \Delta t) \\ \omega \Delta t + \gamma \Delta t \end{pmatrix}$$



Term to account for the final rotation

Examples (Velocity-Based)



Sensor Model

$$bel(x_t) = \eta p(z_t | x_t) \overline{bel}(x_{t-1})$$

Model for Laser Scanners

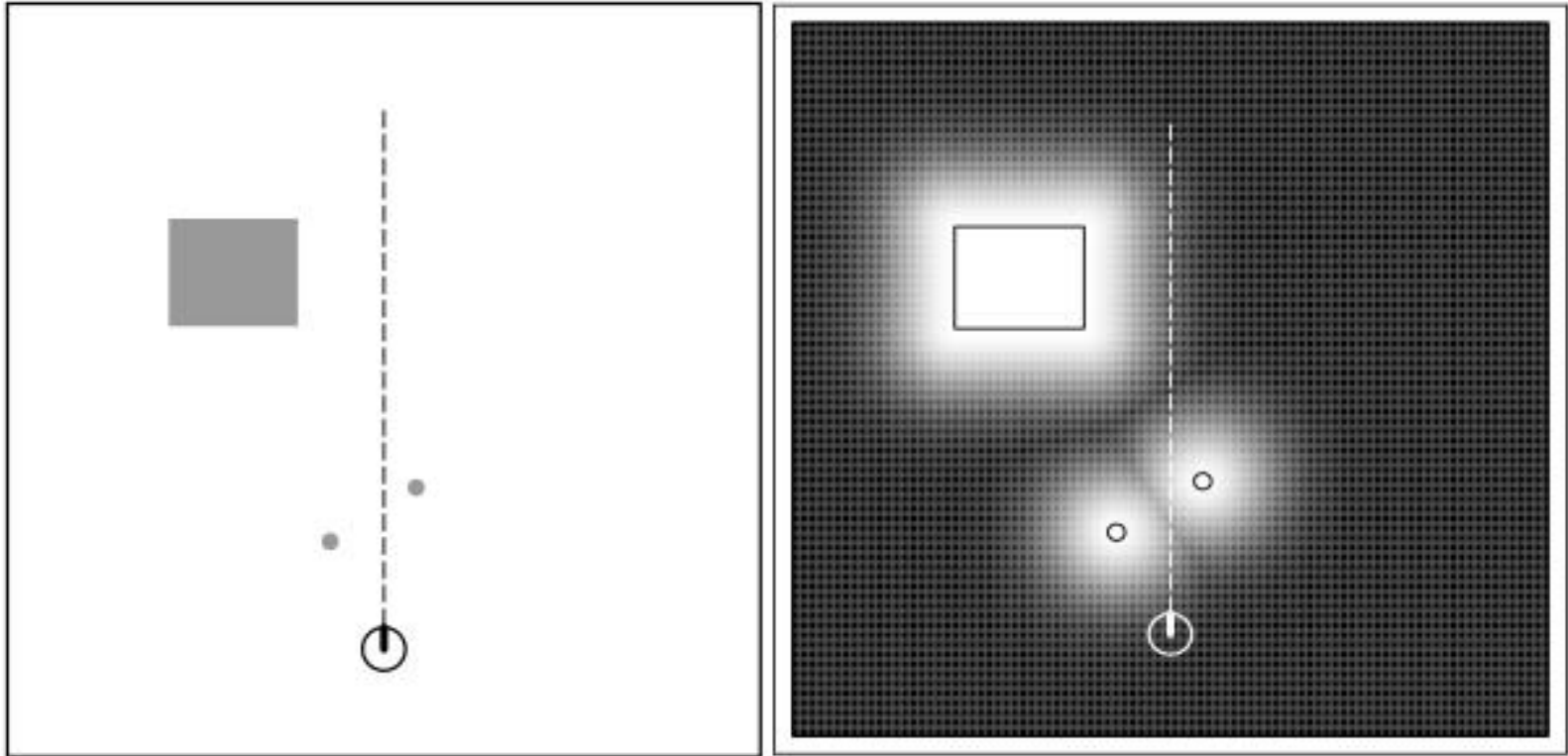
- Scan z consists of K measurements.

$$z_t = \{z_t^1, \dots, z_t^k\}$$

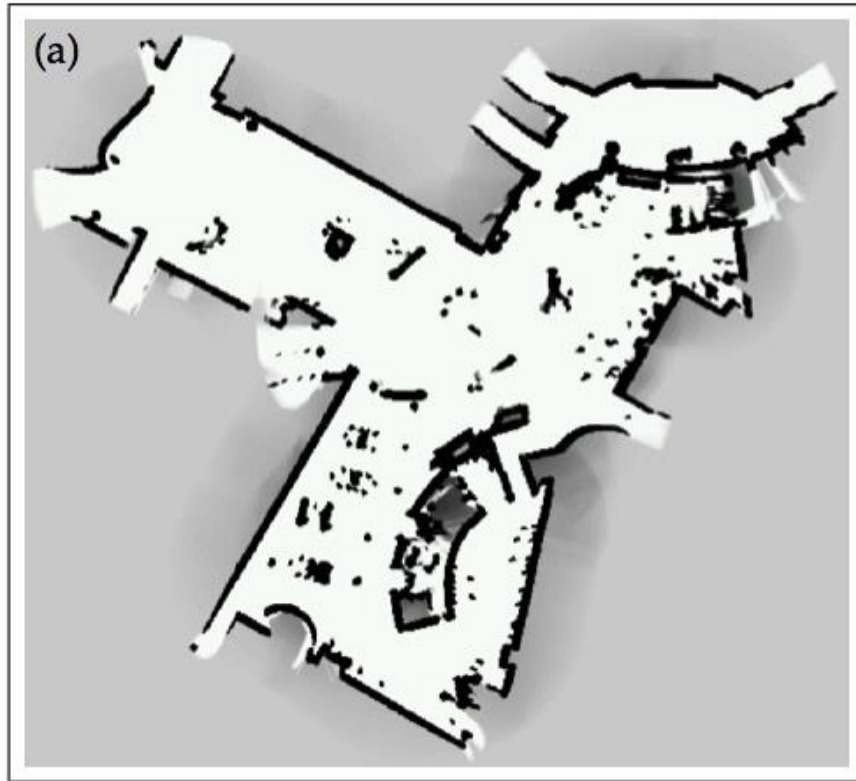
- Individual measurements are independent given the robot position

$$p(z_t \mid x_t, m) = \prod_{i=1}^k p(z_t^i \mid x_t, m)$$

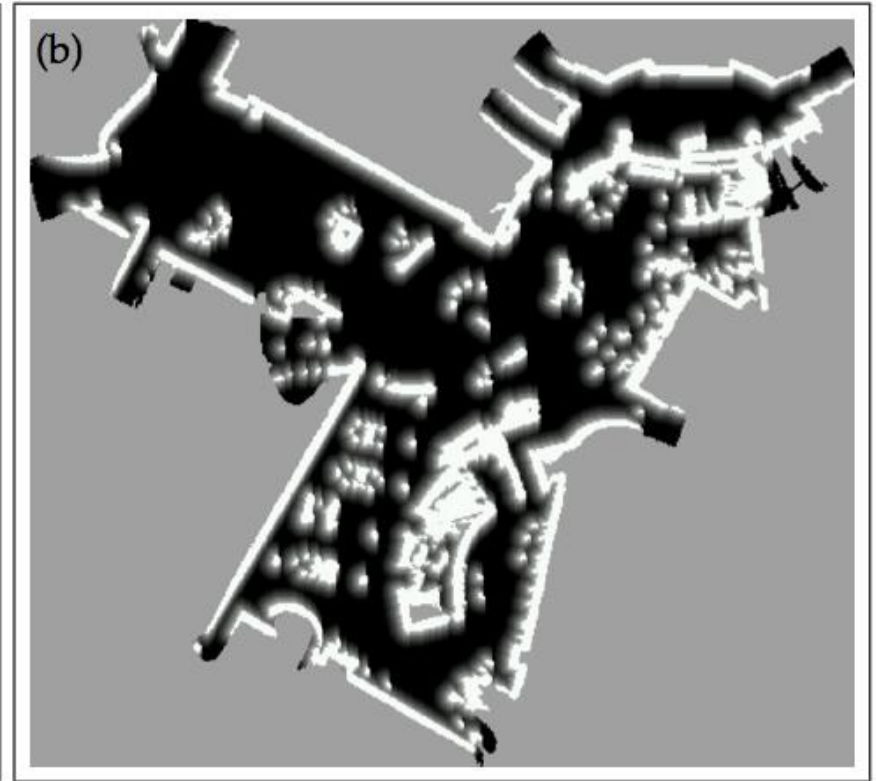
Beam-Endpoint Model



Beam-Endpoint Model



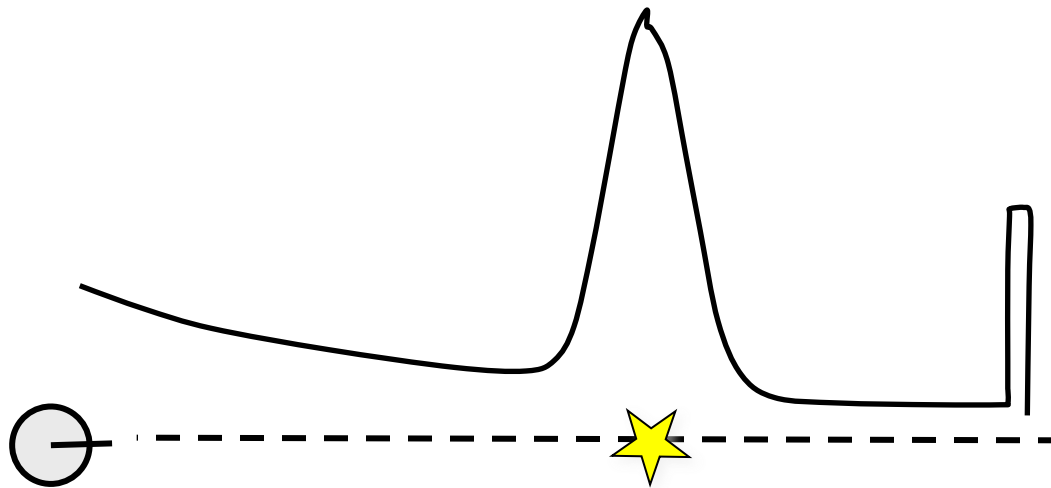
map



likelihood field

Ray-cast Model

- Ray-cast model considers the first obstacle along the line of sight
- Mixture of four models



Model for Perceiving Landmarks with Range-Bearing Sensors

- Range-bearing $z_t^i = (r_t^i, \phi_t^i)^T$
- Robot's pose $(x, y, \theta)^T$
- Observation of feature j at location $(m_{j,x}, m_{j,y})^T$

$$\begin{pmatrix} r_t^i \\ \phi_t^i \end{pmatrix} = \begin{pmatrix} \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} \\ \text{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta \end{pmatrix} + Q_t$$

Summary

- Bayes filter is a framework for state estimation
- Motion and sensor model are the central models in the Bayes filter
- Standard models for robot motion and laser-based range sensing

Literature

On the Bayes filter

- Thrun et al. “Probabilistic Robotics”, Chapter 2
- Course: Introduction to Mobile Robotics, Chapter 5

On motion and observation models

- Thrun et al. “Probabilistic Robotics”, Chapters 5 & 6
- Course: Introduction to Mobile Robotics, Chapters 6 & 7

Slide Information

- These slides have been created by Cyrill Stachniss as part of the robot mapping course taught in 2012/13 and 2013/14. I created this set of slides partially extending existing material of Edwin Olson, Pratik Agarwal, and myself.
- I tried to acknowledge all people that contributed image or video material. In case I missed something, please let me know. If you adapt this course material, please make sure you keep the acknowledgements.
- Feel free to use and change the slides. If you use them, I would appreciate an acknowledgement as well. To satisfy my own curiosity, I appreciate a short email notice in case you use the material in your course.
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