# Notes on Univariate Gaussian Distributions and One-Dimensional Kalman Filters

Gian Diego Tipaldi
Department of Computer Science
University of Freiburg
email:tipaldi@cs.uni-freiburg.de

June 19, 2015

**Abstract**

The goal of this document is to prove some properties of Gaussian distributions and the relationship between the generic Bayes filter with the Kalman filter if the underlying distributions are Gaussians. We limit the analysis on the one dimensional case, as the proofs are shorter. The proofs generalize to the multivariate case by using scalar products and quadratic forms.

## 1 Univariate Gaussian Distributions

The Gaussian (Normal) distribution is a continuous probability distribution with the follow probability density function (pdf):

$$p\left(x;\mu,\sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\}. \tag{1}$$

We then say that a random variable $X$ is normally (Gaussian) distributed and write $X \sim \mathcal{N}\left(\mu,\sigma^2\right)$.

The parameters of the pdf represent the first two moments of the distribution

$$\mu = \mathbb{E}_X[X] \quad = \int_{-\infty}^{\infty} x p\left(x;\mu,\sigma^2\right) \mathrm{d}x \tag{2}$$

$$\sigma^2 = \mathbb{E}_X[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 p\left(x;\mu,\sigma^2\right) \mathrm{d}x \tag{3}$$

### 1.1 Derivation of the Probability Density Function of the Product of two Gaussian

Assume we have two random variables with Gaussian pdf

$$X_1 \sim \mathcal{N}\left(\mu_1,\sigma_1^2\right) \quad \Rightarrow \quad p_1(x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left\{-\frac{1}{2}\frac{(x-\mu_1)^2}{\sigma_1^2}\right\}, \tag{4}$$

$$X_2 \sim \mathcal{N}\left(\mu_2,\sigma_2^2\right) \quad \Rightarrow \quad p_2(x) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left\{-\frac{1}{2}\frac{(x-\mu_2)^2}{\sigma_2^2}\right\}. \tag{5}$$

What is the distribution of the product of the two pdf, $\hat{p}(x) = p_1(x) \cdot p_2(x)$?

Computing the product, we have

$$p_1(x) \cdot p_2(x) = \eta \exp\left\{ -\frac{1}{2} \left( \frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(x-\mu_2)^2}{\sigma_2^2} \right) \right\}, \tag{6}$$

where $\eta$ is a normalization factor to ensure the density sum up to one and is a proper pdf.

If we expand the terms in the exponential part and collect them with respect to $x^2$ and $x$, we have

$$\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(x-\mu_2)^2}{\sigma_2^2} = \tag{7}$$

$$= \frac{\sigma_2^2 \cdot (x-\mu_1)^2 + \sigma_1^2 \cdot (x-\mu_2)^2}{\sigma_1^2 \cdot \sigma_2^2} \tag{8}$$

$$= \frac{\sigma_2^2 x^2 - 2x\sigma_2^2\mu_1 + \sigma_2^2\mu_1^2 + \sigma_1^2 x^2 - 2x\sigma_1^2\mu_2 + \sigma_1^2\mu_2^2}{\sigma_1^2 \cdot \sigma_2^2} \tag{9}$$

$$= \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 \cdot \sigma_2^2} x^2 - 2\frac{\sigma_2^2\mu_1 + \sigma_1^2\mu_2^2}{\sigma_1^2 \cdot \sigma_2^2} x + \frac{\sigma_2^2\mu_1^2 + \sigma_1^2\mu_2^2}{\sigma_1^2 \cdot \sigma_2^2}. \tag{10}$$

Let now consider a generic Gaussian pdf with mean $\hat{\mu}$ and variance $\hat{\sigma}$ and expand the terms in the exponential part as well. We obtain

$$\frac{(x-\hat{\mu})^2}{\hat{\sigma}^2} = \frac{1}{\hat{\sigma}^2} x^2 - 2\frac{\hat{\mu}}{\hat{\sigma}^2} x + \frac{\hat{\mu}^2}{\hat{\sigma}^2} \tag{11}$$

We now match the corresponding terms for $x$ and $x^2$ to obtain the parameters of the new Gaussian. Note that since the pdf is a function of $x$, we can dismiss the terms that do not depend on $x$, since they will be incorporated into the normalization term $\eta$. In practice, we need to solve the system (this time in $\hat{\mu}$ and $\hat{\sigma}$)

$$\frac{1}{\hat{\sigma}^2} = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 \cdot \sigma_2^2} \tag{12}$$

$$\frac{\hat{\mu}}{\hat{\sigma}^2} = \frac{\sigma_2^2\mu_1 + \sigma_1^2\mu_2^2}{\sigma_1^2 \cdot \sigma_2^2}. \tag{13}$$

Solving for the first equation, we have

$$\hat{\sigma}^2 = \frac{\sigma_1^2 \cdot \sigma_2^2}{\sigma_1^2 + \sigma_2^2}. \tag{14}$$

Substituting this result into the second equation, we have

$$\hat{\mu} = \frac{\sigma_2^2\mu_1 + \sigma_1^2\mu_2^2}{\sigma_1^2 \cdot \sigma_2^2} \cdot \hat{\sigma}^2 \tag{15}$$

$$= \frac{\sigma_2^2\mu_1 + \sigma_1^2\mu_2^2}{\sigma_1^2 \cdot \sigma_2^2} \cdot \frac{\sigma_1^2 \cdot \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \tag{16}$$

$$= \frac{\sigma_2^2\mu_1 + \sigma_1^2\mu_2^2}{\sigma_1^2 + \sigma_2^2}. \tag{17}$$

## 1.2 Derivation of the Probability Density Function of the Convolution of two Gaussian

Assume we have two random variables, $X$ and $Y$ and we know that

$$X \sim \mathcal{N}\left(\mu_X, \sigma_X^2\right) \quad \Rightarrow \quad p\left(x\right) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left\{-\frac{1}{2}\frac{\left(x-\mu_X\right)^2}{\sigma_X^2}\right\}, \tag{18}$$

$$Y|x \sim \mathcal{N}\left(ax+b, \sigma_r^2\right) \Rightarrow p\left(y \mid x\right) = \frac{1}{\sqrt{2\pi\sigma_r^2}} \exp\left\{-\frac{1}{2}\frac{\left(y-ax-b\right)^2}{\sigma_r^2}\right\}, \tag{19}$$

i.e., $Y$ is obtained from an affine transformation of $x$. What is the distribution of $Y$? According to the law of total probability, we need to compute the following integral

$$p(y) = \int_{-\infty}^{\infty} p(y|x)p(x)\mathrm{d}x \tag{20}$$

If we know that the the distribution of $Y$ is Gaussian, we could limit ourselves to compute the moments of the distribution, without explicitly solve the integral. We have, for the mean

$$\begin{aligned}
\mu_Y = \mathbb{E}_Y[Y] &= \int_{-\infty}^{\infty} y \left(\int_{-\infty}^{\infty} p(y|x)p(x)\mathrm{d}x\right)\mathrm{d}y && (21)\\
&= \int_{-\infty}^{\infty} p(x)\left(\int_{-\infty}^{\infty} yp(y|x)\mathrm{d}y\right)\mathrm{d}x && (22)\\
&= \int_{-\infty}^{\infty} \mathbb{E}_{Y|x}[Y]p(x)\mathrm{d}x && (23)\\
&= \int_{-\infty}^{\infty} (ax+b)p(x)\mathrm{d}x && (24)\\
&= a\int_{-\infty}^{\infty} xp(x)\mathrm{d}x + b && (25)\\
&= a\ \mathbb{E}_X[X] + b && (26)\\
&= a\mu_X + b, && (27)
\end{aligned}$$

where we used the relation between the expected value, its integral and the mean of a Gaussian distribution.

In a similar way, we can compute the variance. Let first recall the relation

$$\begin{aligned}
\mathbb{E}[(X-\mathbb{E}[X])^2] &= \mathbb{E}[X^2 + \mathbb{E}[X]^2 - 2X\mathbb{E}[X]] && (28)\\
&= \mathbb{E}[X^2] + \mathbb{E}[X]^2 - 2\mathbb{E}[X]\mathbb{E}[X] && (29)\\
&= \mathbb{E}[X^2] - \mathbb{E}[X]^2. && (30)
\end{aligned}$$

Exploiting that and the definition of variance, we can compute

$$
\begin{aligned}
\sigma_Y^2 &= \mathbb{E}_Y[(Y - \mathbb{E}_Y[Y])^2] = \mathbb{E}_Y[Y^2] - \mathbb{E}_Y[Y]^2 && (31)\\
&= \int_{-\infty}^{\infty} y^2 \left( \int_{-\infty}^{\infty} p(y|x)p(x)\mathrm{d}x \right) \mathrm{d}y - \mathbb{E}_Y[Y]^2 && (32)\\
&= \int_{-\infty}^{\infty} p(x) \left( \int_{-\infty}^{\infty} y^2 p(y|x)\mathrm{d}y \right) \mathrm{d}x - \mathbb{E}_Y[Y]^2 && (33)\\
&= \int_{-\infty}^{\infty} p(x) \left( \mathbb{E}_{Y|x}[Y^2] \right) \mathrm{d}x - \mathbb{E}_Y[Y]^2 && (34)\\
&= \int_{-\infty}^{\infty} p(x) \left( \mathbb{E}_{Y|x}[(Y - \mathbb{E}_{Y|x}[Y])^2] + \mathbb{E}_{Y|x}[Y]^2 \right) \mathrm{d}x - \mathbb{E}_Y[Y]^2 && (35)\\
&= \int_{-\infty}^{\infty} p(x) \left( \sigma_r^2 + (ax+b)^2 \right) \mathrm{d}x - \mathbb{E}_Y[Y]^2 && (36)\\
&= \sigma_r^2 + \int_{-\infty}^{\infty} (ax+b)^2 p(x)\mathrm{d}x - \mathbb{E}_Y[Y]^2 && (37)\\
&= \sigma_r^2 + \mathbb{E}_X[(aX+b)^2] - \mathbb{E}_Y[Y]^2 && (38)\\
&= \sigma_r^2 + \mathbb{E}_X[(aX+b-\mathbb{E}_X[(aX+b)])^2] + \mathbb{E}_X[(aX+b)]^2 - \mathbb{E}_Y[Y]^2 && (39)\\
&= \sigma_r^2 + \mathbb{E}_X[(aX+b-a\,\mathbb{E}_X[X]-b)^2] + (a\mu_X+b)^2 - \mathbb{E}_Y[Y]^2 && (40)\\
&= \sigma_r^2 + a^2\mathbb{E}_X[(X-\mathbb{E}_X[X])^2] + (a\mu_X+b)^2 - (a\mu_X+b)^2 && (41)\\
&= \sigma_r^2 + a^2\sigma_X^2. && (42)
\end{aligned}
$$

In general, however, we might not know that the distribution is Gaussian and computing the first two moments does not tell us which kind of distribution it is. In this case, we need to compute the full integral. To do that, we use two tricks. The first is the solution of this integral

$$
\int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2} \right\} \mathrm{d}x = \sqrt{2\pi\sigma^2}. \quad (43)
$$

The second is the trick of *completing the square* to bring ourselves to that integral form.

With those two tricks, we can finally compute the integral

$$
\begin{aligned}
p(y) &= \int_{-\infty}^{\infty} p(y|x)p(x)\mathrm{d}x && (44)\\
&= \eta \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2}\frac{(x-\mu_X)^2}{\sigma_X^2} \right\} \exp\left\{ -\frac{1}{2}\frac{(y-ax-b)^2}{\sigma_r^2} \right\} \mathrm{d}x, && (45)\\
&= \eta \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2}\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{1}{2}\frac{(y-ax-b)^2}{\sigma_r^2} \right\} \mathrm{d}x, && (46)
\end{aligned}
$$

where we collected all normalization terms in $\eta$. Ideally, we would like to decompose the exponent in two terms: one that depends on $x$ and can be written in the squared form for which we can compute the integral, and one which does not and we can put outside the the integral. In practice, we want to find $L_x$

and $L_y$, such that

$$L = L_x + L_y = \frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-ax-b)^2}{\sigma_r^2} \tag{47}$$

$$L_x = \frac{(x-\beta)^2}{\alpha^2} \tag{48}$$

To do so, we need to complete the square. Do you remember the rules for solving a quadratic equation? We will use the same one. We want to transform

$$ax^2 + bx + c \tag{49}$$

in something like

$$\frac{1}{\alpha^2}(x-\beta)^2 + \kappa. \tag{50}$$

By matching the coefficients we have

$$\alpha^2 = \frac{1}{a}; \qquad \beta = -\frac{b}{2a}; \qquad \kappa = c - \frac{b^2}{4a}. \tag{51}$$

Let now expand the exponential term to bring ourselves in the first situation[1].

$$L = \sigma_X^{-2}x^2 - 2\sigma_X^{-2}x\mu_X + \sigma_X^{-2}\mu_X^2 + \sigma_r^{-2}a^2x^2 - 2ax(y-b) + \sigma_r^{-2}(y-b)^2 \tag{52}$$

$$= (\sigma_X^{-2} + \sigma_r^{-2}a^2)x^2 - 2(\sigma_X^{-2}\mu_X + \sigma_r^{-2}a(y-b))x + \sigma_X^{-2}\mu^2 + \sigma_r^{-2}(y-b)^2 \tag{53}$$

$$= \underbrace{(\sigma_X^{-2} + \sigma_r^{-2}a^2)\left(x - \frac{\sigma_X^{-2}\mu_X + \sigma_r^{-2}a(y-b)}{\sigma_X^{-2} + \sigma_r^{-2}a^2}\right)^2}_{L_x} +$$

$$+ \underbrace{\sigma_X^{-2}\mu^2 + \sigma_r^{-2}(y-b)^2 - \frac{(\sigma_X^{-2}\mu_X + \sigma_r^{-2}a(y-b))^2}{\sigma_X^{-2} + \sigma_r^{-2}a^2}}_{L_y}. \tag{54}$$

Let now simplify $L_y$. Expanding the squares and rearranging terms we obtain

$$L_y = \frac{(\sigma_X^{-2} + \sigma_r^{-2}a^2)\sigma_X^{-2}\mu_X^2 + (\sigma_X^{-2} + \sigma_r^{-2}a^2)\sigma_r^{-2}(y-b)^2}{\sigma_X^{-2} + \sigma_r^{-2}a^2} +$$

$$+ \frac{-\sigma_X^{-4}\mu_X^2 - \sigma_r^{-4}a^2(y-b)^2 - 2\sigma_X^{-2}\mu_X\sigma_r^{-2}a(y-b)}{\sigma_X^{-2} + \sigma_r^{-2}a^2} \tag{55}$$

$$= \frac{\sigma_r^{-2}\sigma_X^{-2}(a^2\mu_X^2 + (y-b)^2 - 2a\mu_X(y-b))}{\sigma_X^{-2} + \sigma_r^{-2}a^2} \tag{56}$$

$$= \frac{\sigma_r^{-2}\sigma_X^{-2}(y - a\mu_X - b)^2}{\sigma_X^{-2} + \sigma_r^{-2}a^2} \tag{57}$$

[1] For simplicity I will write $\sigma^{-2}$ instead of $\frac{1}{\sigma^2}$

Putting everything together, we have

$$
\begin{aligned}
p(y) &= \int_{-\infty}^{\infty} p(y|x)p(x)\mathrm{d}x && (58)\\
&= \frac{1}{\sqrt{2\pi\sigma_X^2}}\frac{1}{\sqrt{2\pi\sigma_r^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}(L_x+L_y)\right\}\mathrm{d}x && (59)\\
&= \frac{1}{\sqrt{2\pi\sigma_X^2}}\frac{1}{\sqrt{2\pi\sigma_r^2}}\exp\left\{-\frac{1}{2}L_y\right\}\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}L_x\right\}\mathrm{d}x && (60)\\
&= \frac{1}{\sqrt{2\pi\sigma_X^2}}\frac{1}{\sqrt{2\pi\sigma_r^2}}\sqrt{2\pi(\sigma_X^{-2}+\sigma_r^{-2}a^2)}\exp\left\{-\frac{1}{2}L_y\right\} && (61)\\
&= \frac{\sqrt{\sigma_X^{-2}+\sigma_r^{-2}a^2}}{\sqrt{2\pi\sigma_X^2\sigma_r^2}}\exp\left\{-\frac{1}{2}\frac{\sigma_r^{-2}\sigma_X^{-2}(y-a\mu_X-b)^2}{\sigma_X^{-2}+\sigma_r^{-2}a^2}\right\} && (62)\\
&= \frac{1}{\sqrt{2\pi\sigma_Y}}\exp\left\{-\frac{1}{2}\frac{(y-\mu_Y)^2}{\sigma_Y^{-2}}\right\}, && (63)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_Y &= a\mu_X + b && (64)\\
\sigma_Y^2 &= \frac{\frac{1}{\sigma_X^2}+\frac{a^2}{\sigma_r^2}}{\frac{1}{\sigma_r^2\sigma_X^2}} = \sigma_r^2 + a^2\sigma_X^2 && (65)
\end{aligned}
$$

This shows that the resulting distribution is Gaussian with mean $\mu_Y = a\mu_X + b$ and variance $\sigma_Y^2 = \sigma_r^2 + a^2\sigma_X^2$

# 2 One-Dimensional Kalman Filter

Consider the following one-dimensional linear system

$$
\begin{aligned}
x_t &= a_t x_{t-1} + b_t u_t + \epsilon_t && (66)\\
z_t &= c_t x_t + \delta_t, && (67)
\end{aligned}
$$

where

$$
\epsilon_t \sim \mathcal{N}\left(0, \sigma_{Q,t}^2\right), \qquad \delta_t \sim \mathcal{N}\left(0, \sigma_{R,t}^2\right). \tag{68}
$$

Let also assume that the belief about the initial state is Gaussian $x_0 \sim \mathcal{N}\left(\mu_0, \sigma_0^2\right)$.

In this probabilistic setting, the Kalman filter is an implementation of the Bayes filter when all the distributions are Gaussians and the observation and motion models are linear. The Kalman filter keeps track of the mean and the variance of the filtering density during the predict and update cycle. Since the density is Gaussian, those two statistics are sufficient and therefore the Kalman filter is optimal for the linear-Gaussian case.

In the remainder of this section we will show how to derive the Kalman filter equation from the generic framework of the Bayes filter.

## 2.1 Predict Step

In the prediction step of the Bayes filter we have

$$\overline{bel}(x_t) = \int_{-\infty}^{\infty} p(x_t|u_t, x_{t-1})\, bel(x_{t-1}) \mathrm{d}x_{t-1}. \tag{69}$$

From the recursive step and the linear Gaussian system, we also have that

$$bel(x_{t-1}) \sim \mathcal{N}\left(\mu_{t-1}, \sigma_{t-1}^2\right) \tag{70}$$
$$p(x_t|u_t, x_{t-1}) \sim \mathcal{N}\left(a_t x_{t-1} + b_t u_t, \sigma_{Q,t}^2\right). \tag{71}$$

Following the derivation we did in Section 1.2 and using the results of (64) and (65), we have that the resulting distribution is still a Gaussian distribution

$$\overline{bel}(x_t) \sim \mathcal{N}\left(\overline{\mu}_t, \overline{\sigma}_t^2\right) \tag{72}$$
$$\overline{\mu}_t = a_t \mu_{t-1} + b_t u_t \tag{73}$$
$$\overline{\sigma}_t^2 = a_t^2 \sigma_{t-1}^2 + \sigma_{Q,t}^2. \tag{74}$$

Note that the last two equations are exaxtly the prediction equations of the Kalman filter.

## 2.2 Update Step

In the correction step of the Bayes filter we have

$$bel(x_t) = \eta p(z_t|x_t)\overline{bel}(x_t). \tag{75}$$

From the update step (previous subsection) and the linear-Gaussian observation, we also have that

$$\overline{bel}(x_t) \sim \mathcal{N}\left(\overline{\mu}_t, \overline{\sigma}_t^2\right) \tag{76}$$
$$p(z_t|x_t) \sim \mathcal{N}\left(c_t x_t, \sigma_{R,t}^2\right). \tag{77}$$

Expanding the exponents of the product and rearranging terms, we have

$$\frac{(z_t - c_t x_t)^2}{\sigma_{R,t}^2} + \frac{(x_t - \overline{\mu}_t)^2}{\overline{\sigma}_t^2} = \tag{78}$$
$$= \frac{\overline{\sigma}_t^2 c_t^2 + \sigma_{R,t}^2}{\overline{\sigma}_t^2 \cdot \sigma_{R,t}^2} x^2 - 2\frac{\sigma_{R,t}^2 \overline{\mu}_t + \overline{\sigma}_t^2 c_t z_t}{\overline{\sigma}_t^2 \cdot \sigma_{R,t}^2} x + \frac{\sigma_{R,t}^2 \overline{\mu}_t^2 + \overline{\sigma}_t^2 z_t^2}{\overline{\sigma}_t^2 \cdot \sigma_{R,t}^2}. \tag{79}$$

The expression is the same we had in Section 1.1. Using the results of (17) and (14), we have that the resulting distribution is still a Gaussian

$$bel(x_t) \sim \mathcal{N}\left(\mu_t, \sigma_t^2\right) \tag{80}$$
$$\mu_t = \frac{\sigma_{R,t}^2 \overline{\mu}_t + \overline{\sigma}_t^2 c_t z_t}{\overline{\sigma}_t^2 c_t^2 + \sigma_{R,t}^2} \tag{81}$$
$$\sigma_t^2 = \frac{\overline{\sigma}_t^2 \cdot \sigma_{R,t}^2}{\overline{\sigma}_t^2 c_t^2 + \sigma_{R,t}^2}. \tag{82}$$

Those two equations, however, look different than the Kalman filter one. Where does the Kalman gain coming from? To obtain the standard form of the Kalman filter, we need to rearrange few terms. Let's do it for the mean

$$\mu_t = \frac{\sigma^2_{R,t}\overline{\mu}_t + \overline{\sigma}^2_t c_t z_t}{\overline{\sigma}^2_t c^2_t + \sigma^2_{R,t}} \tag{83}$$

$$= \frac{\sigma^2_{R,t}\overline{\mu}_t + \overline{\sigma}^2_t c_t z_t + \overline{\sigma}^2_t c^2_t \overline{\mu}_t - \overline{\sigma}^2_t c^2_t \overline{\mu}_t}{\overline{\sigma}^2_t c^2_t + \sigma^2_{R,t}} \tag{84}$$

$$= \frac{\overline{\sigma}^2_t c^2_t + \sigma^2_{R,t}}{\overline{\sigma}^2_t c^2_t + \sigma^2_{R,t}}\overline{\mu}_t + \frac{\overline{\sigma}^2_t c_t}{\overline{\sigma}^2_t c^2_t + \sigma^2_{R,t}}(z_t - c_t\overline{\mu}_t) \tag{85}$$

$$= \overline{\mu}_t + K_t(z_t - c_t\overline{\mu}_t) \tag{86}$$

$$K_t = \frac{\overline{\sigma}^2_t c_t}{\overline{\sigma}^2_t c^2_t + \sigma^2_{R,t}} \tag{87}$$

Similarly, one can do the same trick for the variance

$$\sigma^2_t = \frac{\overline{\sigma}^2_t \cdot \sigma^2_{R,t}}{\overline{\sigma}^2_t c^2_t + \sigma^2_{R,t}} \tag{88}$$

$$= \frac{\overline{\sigma}^2_t \cdot \sigma^2_{R,t} + \overline{\sigma}^4_t c^2_t - \overline{\sigma}^4_t c^2_t}{\overline{\sigma}^2_t c^2_t + \sigma^2_{R,t}} \tag{89}$$

$$= \frac{\sigma^2_{R,t} + \overline{\sigma}^2_t c^2_t}{\overline{\sigma}^2_t c^2_t + \sigma^2_{R,t}}\overline{\sigma}^2_t - \frac{\overline{\sigma}^2_t c^2_t}{\overline{\sigma}^2_t c^2_t + \sigma^2_{R,t}}\overline{\sigma}^2_t \tag{90}$$

$$= \overline{\sigma}^2_t - K_t c_t \overline{\sigma}^2_t = (1 - K_t c_t)\overline{\sigma}^2_t. \tag{91}$$

$$K_t = \frac{\overline{\sigma}^2_t c_t}{\overline{\sigma}^2_t c^2_t + \sigma^2_{R,t}} \tag{92}$$

A similar proof can be done for multivariate distributions, replacing squares with symmetric products of matrices, fractions with matrix inversion and keeping track of when a matrix must be transpose.