Robot Mapping

A Short Introduction to the Bayes Filter and Related Models

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State Estimation

- Estimate the state $x$ of a system given observations $z$ and controls $u$

- Goal:

$$p(x \mid z, u)$$
Recursive Bayes Filter 1

$bel(x_t) = p(x_t | z_{1:t}, u_{1:t})$

Definition of the belief
Recursive Bayes Filter 2

\[
\text{bel}(x_t) = p(x_t \mid z_{1:t}, u_{1:t}) \\
= \eta p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) p(x_t \mid z_{1:t-1}, u_{1:t})
\]

Bayes’ rule
Recursive Bayes Filter 3

\[
bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t})
= \eta p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) p(x_t \mid z_{1:t-1}, u_{1:t})
= \eta p(z_t \mid x_t) p(x_t \mid z_{1:t-1}, u_{1:t})
\]

Markov assumption
Recursive Bayes Filter 4

\[
\text{bel}(x_t) = p(x_t \mid z_{1:t}, u_{1:t}) \\
= \eta \ p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) \ p(x_t \mid z_{1:t-1}, u_{1:t}) \\
= \eta \ p(z_t \mid x_t) \ p(x_t \mid z_{1:t-1}, u_{1:t}) \\
= \eta \ p(z_t \mid x_t) \int \ p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \ p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \ dx_{t-1}
\]

Law of total probability
Recursive Bayes Filter 5

\[ bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t}) \]

\[ = \eta p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) p(x_t \mid z_{1:t-1}, u_{1:t}) \]

\[ = \eta p(z_t \mid x_t) p(x_t \mid z_{1:t-1}, u_{1:t}) \]

\[ = \eta p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \, dx_{t-1} \]

\[ = \eta p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t) p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \, dx_{t-1} \]

Markov assumption
Recursive Bayes Filter 6

\[ \text{bel}(x_t) = p(x_t \mid z_{1:t}, u_{1:t}) \]

\[ = \eta \ p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) \ p(x_t \mid z_{1:t-1}, u_{1:t}) \]

\[ = \eta \ p(z_t \mid x_t) \ p(x_t \mid z_{1:t-1}, u_{1:t}) \]

\[ = \eta \ p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \]

\[ p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \ dx_{t-1} \]

\[ = \eta \ p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t) \ p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \ dx_{t-1} \]

\[ = \eta \ p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t) \ p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1}) \ dx_{t-1} \]

Markov assumption
Recursive Bayes Filter 7

\[ bel(x_t) = p(x_t \mid z_{1:t}, u_{1:t}) \]

\[ = \eta \ p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) \ p(x_t \mid z_{1:t-1}, u_{1:t}) \]

\[ = \eta \ p(z_t \mid x_t) \ p(x_t \mid z_{1:t-1}, u_{1:t}) \]

\[ = \eta \ p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \]

\[ p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \ dx_{t-1} \]

\[ = \eta \ p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t) \ p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) \ dx_{t-1} \]

\[ = \eta \ p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t) \ p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1}) \ dx_{t-1} \]

\[ = \eta \ p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t) \ bel(x_{t-1}) \ dx_{t-1} \]

Recursive term
Prediction and Correction Step

- Bayes filter can be written as a two step process

  - **Prediction step**
    \[
    \overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \, bel(x_{t-1}) \, dx_{t-1}
    \]

  - **Correction step**
    \[
    bel(x_t) = \eta \, p(z_t \mid x_t) \, \overline{bel}(x_t)
    \]
Motion and Observation Model

- **Prediction step**

  \[
  \overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) \overline{bel}(x_{t-1}) \, dx_{t-1}
  \]

  **motion model**

- **Correction step**

  \[
  bel(x_t) = \eta \, p(z_t | x_t) \, \overline{bel}(x_t)
  \]

  **sensor or observation model**
Different Realizations

▪ The Bayes filter is a **framework** for recursive state estimation
▪ There are **different realizations**
▪ **Different properties**
  ▪ Linear vs. non-linear models for motion and observation models
  ▪ Gaussian distributions only?
  ▪ Parametric vs. non-parametric filters
  ▪ ...
In this Course

- Kalman filter & friends
  - Gaussians
  - Linear or linearized models

- Particle filter
  - Non-parametric
  - Arbitrary models (sampling required)
\[ \overline{\text{bel}}(x_t) = \int \underbrace{p(x_t \mid u_t, x_{t-1}) \text{bel}(x_{t-1})} \, dx_{t-1} \]
Robot Motion Models

- Robot motion is inherently uncertain
- How can we model this uncertainty?

Courtesy: Thrun, Burgard, Fox
Probabilistic Motion Models

- Specifies a posterior probability that action \( u \) carries the robot from \( x \) to \( x' \).

\[
p(x_t \mid u_t, x_{t-1})
\]
Typical Motion Models

- In practice, one often finds two types of motion models:
  - Odometry-based
  - Velocity-based
- Odometry-based models for systems that are equipped with wheel encoders
- Velocity-based when no wheel encoders are available
Odometry Model

- Robot moves from \((\bar{x}, \bar{y}, \bar{\theta})\) to \((\bar{x}', \bar{y}', \bar{\theta}')\)
- Odometry information \( u = (\delta_{rot1}, \delta_{trans}, \delta_{rot2}) \)

\[
\delta_{trans} = \sqrt{(x' - x)^2 + (y' - y)^2}
\]
\[
\delta_{rot1} = \text{atan2}(y' - y, x' - x) - \bar{\theta}
\]
\[
\delta_{rot2} = \bar{\theta}' - \bar{\theta} - \delta_{rot1}
\]
Probability Distribution

- Noise in odometry \( u = (\delta_{\text{rot1}}, \delta_{\text{trans}}, \delta_{\text{rot2}}) \)
- Example: Gaussian noise

\[ u \sim \mathcal{N}(0, \Sigma) \]
Examples (Odometry-Based)

Courtesy: Thrun, Burgard, Fox
Velocity-Based Model

\[ u = (v, \omega)^T \]

\[ \theta - 90 \]

\[ r \]

\[ \theta \]
Motion Equation

- Robot moves from \((x, y, \theta)\) to \((x', y', \theta')\)
- Velocity information \(u = (v, \omega)\)

\[
\begin{pmatrix}
  x' \\
  y' \\
  \theta'
\end{pmatrix}
= \begin{pmatrix}
  x \\
  y \\
  \theta
\end{pmatrix}
+ \begin{pmatrix}
  -\frac{v}{\omega} \sin \theta + \frac{v}{\omega} \sin(\theta + \omega \Delta t) \\
  \frac{v}{\omega} \cos \theta - \frac{v}{\omega} \cos(\theta + \omega \Delta t) \\
  \omega \Delta t
\end{pmatrix}
\]
Problem of the Velocity-Based Model

- Robot moves on a circle
- The circle constrains the final orientation
- **Fix:** introduce an additional noise term on the final orientation
Motion Including 3\textsuperscript{rd} Parameter

\[
\begin{pmatrix}
  x' \\
  y' \\
  \theta'
\end{pmatrix}
= \begin{pmatrix}
  x \\
  y \\
  \theta
\end{pmatrix} + \begin{pmatrix}
  -\frac{v}{\omega} \sin \theta + \frac{v}{\omega} \sin(\theta + \omega \Delta t) \\
  \frac{v}{\omega} \cos \theta - \frac{v}{\omega} \cos(\theta + \omega \Delta t) \\
  \omega \Delta t + \gamma \Delta t
\end{pmatrix}
\]

Term to account for the final rotation
Examples (Velocity-Based)

Courtesy: Thrun, Burgard, Fox
Sensor Model

\[ \text{bel}(x_t) = \eta p(z_t | x_t) \text{bel}(x_{t-1}) \]
Model for Laser Scanners

- Scan $z$ consists of $K$ measurements.

$$z_t = \{z_t^1, \ldots, z_t^k\}$$

- Individual measurements are independent given the robot position

$$p(z_t \mid x_t, m) = \prod_{i=1}^{k} p(z_t^i \mid x_t, m)$$
Beam-Endpoint Model

Courtesy: Thrun, Burgard, Fox
Beam-Endpoint Model

(a) map

(b) likelihood field

Courtesy: N. Roy
Ray-cast Model

- Ray-cast model considers the first obstacle along the line of sight
- Mixture of four models
Model for Perceiving Landmarks with Range-Bearing Sensors

- Range-bearing $z^i_t = (r^i_t, \phi^i_t)^T$
- Robot’s pose $(x, y, \theta)^T$
- Observation of feature $j$ at location $(m_j,x, m_j,y)^T$

\[
\begin{pmatrix}
    r^i_t \\
    \phi^i_t
\end{pmatrix} = \begin{pmatrix}
    \sqrt{(m_j,x - x)^2 + (m_j,y - y)^2} \\
    \text{atan2}(m_j,y - y, m_j,x - x) - \theta
\end{pmatrix} + Q_t
\]
Summary

- Bayes filter is a framework for state estimation
- Motion and sensor model are the central models in the Bayes filter
- Standard models for robot motion and laser-based range sensing
Literature

On the Bayes filter

- Thrun et al. “Probabilistic Robotics”, Chapter 2
- Course: Introduction to Mobile Robotics, Chapter 5

On motion and observation models

- Thrun et al. “Probabilistic Robotics”, Chapters 5 & 6
- Course: Introduction to Mobile Robotics, Chapters 6 & 7
Slide Information

- These slides have been created by Cyrill Stachniss as part of the robot mapping course taught in 2012/13 and 2013/14. I created this set of slides partially extending existing material of Edwin Olson, Pratik Agarwal, and myself.

- I tried to acknowledge all people that contributed image or video material. In case I missed something, please let me know. If you adapt this course material, please make sure you keep the acknowledgements.

- Feel free to use and change the slides. If you use them, I would appreciate an acknowledgement as well. To satisfy my own curiosity, I appreciate a short email notice in case you use the material in your course.

- My video recordings are available through YouTube: http://www.youtube.com/playlist?list=PLgnQpQtFTOGQrZ4O5QzbIHgI3b1JHimN_&feature=g-list

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